EXPERIMENTAL GEOMETRY

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INSTRUMENTS

A STRAIGHT-EDGE graduated in inches and tenths, and in centimetres and millimetres.

Pencil compasses.

Set-squares of 45° and 60°.

Protractor.

Scissors.

The above instruments should be in the hands of each student, and the teacher should keep in the school models of the various solid figures and their sections mentioned in the introductory chapter.

The instruments can be had from the publishers.

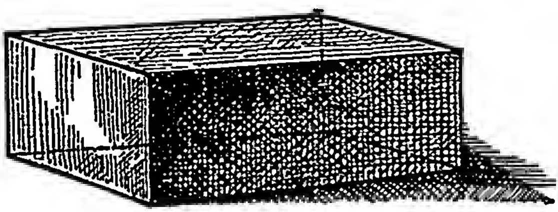
BOOK I

CHAPTER I

INTRODUCTORY

1. LET us examine a rectangular block of wood. We notice that it is solid, has length, breadth, and thickness, and occupies a certain amount of space called its volume. Its bounding faces are six flat surfaces which separate the space it occupies from that which it does not occupy.

Then consider one of the faces. It is a flat surface, so



that if we place the straight edge of a ruler against it in all positions it is in contact with it everywhere. Such surfaces are called plane. Each face is bounded by four lines, and has length and breadth, but no thickness.

Next we have the edges of the solid. These are twelve in number, and are formed by the meeting of the plane surfaces of its faces. These edges are straight, a fact which

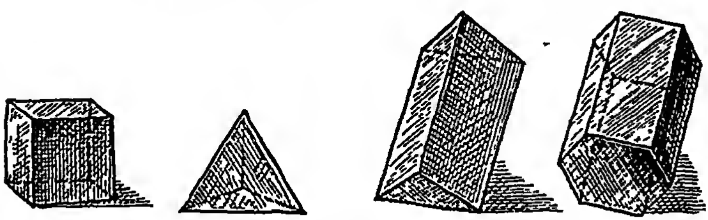
can be tested by applying the ruler to them. The edges have length only.

Lastly, the corners of the solid are eight points, which are formed by the intersections of the lines of its edges. These points have no size, but indicate an exact position.

Each face of this solid is called a rectangle. Faces which are opposite to each other are equal, so that the six faces can be arranged in three sets, each set containing two rectangles exactly like each other. In the same way the edges can be arranged in three sets, each containing four lines of equal length. These facts ought to be tested by actual measurement.

The student ought also to verify that there are three different ways in which we can slice up the whole solid into rectangles of very small thickness, and of size exactly equal to that of one of the faces. He will find on measuring that in all these rectangles the opposite sides are equal, and that the distances between opposite corners, *i.e.* the diagonals, are also equal.

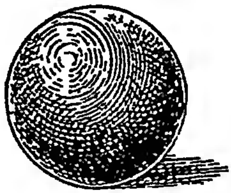
In the manner indicated above, the student ought to study the cube, the regular tetrahedron, the triangular and hexagonal prisms,

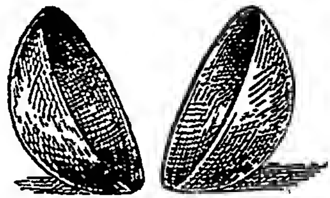


and other solids bounded by plane surfaces. In particular, he ought to make himself familiar with the shapes and names of their various sections.

2. As another example of a solid let us examine a ball

or sphere. It is bounded by a single curved surface. If we cut it into two equal halves we obtain two other solids called hemispheres. Each hemisphere is bounded by a

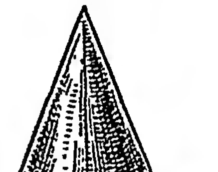


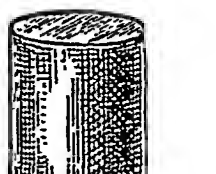


plane surface and a curved surface. The intersection of these two surfaces forms a closed curved line called a circle.

The whole sphere can be cut up into thin circular slices of various sizes, but the largest slice is the one which divides it into two equal halves.

The student ought to examine and name the sections of a right circular cone and cylinder.



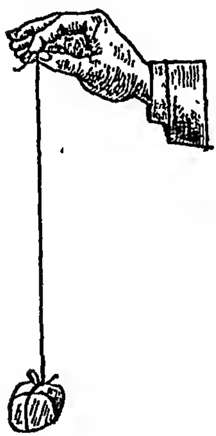


3. From the above we see that solids are bounded by surfaces which may be either plane or curved. When a surface is plane, the straight edge of the ruler placed against it will lie in complete contact with it everywhere, but not when it is curved. With reference to this statement, examine the cone and cylinder.

Surfaces are bounded by lines, and the extremities of lines are points. Two plane surfaces intersect in a straight

line, but the intersection of a plane and a curved surface is a curved line.

Lines, whether straight or curved, intersect in points.



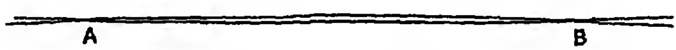
A solid has length, breadth, and thickness; a surface has length and breadth; a line has length only; and a point has no dimensions, but position only.

The Straight Line

4. A very fine thread with a weight suspended at the end will represent a **straight line.**

On paper a straight line is drawn by keeping the fine point of the pencil close to the straight edge of the ruler.

A geometrical line has no breadth; hence its representative on paper should be drawn as fine as possible. In order to test your straight edge, rule a line between the two points *A* and *B*, placing the ruler below them; then rule another line between the same points, and along the same



edge, placing the ruler above the points. If the two lines coincide throughout their entire length, the edge is straight, but if they enclose a space, as in the figure, the edge is not straight.

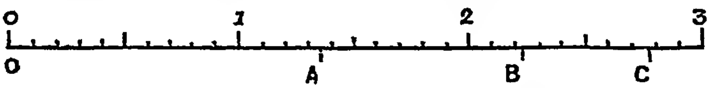
5. **Measurement of Straight Lines.**—Suppose we are required to find the length of the line *AB*. Place one point of the compasses at the extremity *A* of the line, and open out the compasses until the other point rests upon

the extremity B; then the distance between the two points is equal to the length of the line. Now transfer the

open compasses to the graduations on your ruler, and read off the length in inches or centimetres, as may be required.

On the other hand, we may be required to cut off from a given straight line any required length. In that case the above process is reversed. First place the compasses on the ruler, and open them out until the two points are the required distance apart; then transfer the open compasses to the given line, place the needle point at one extremity, and cut off with the pencil point the required distance.

In reading distances on the graduated scale of a ruler, when the point of the compasses does not fall exactly on a



subdivision mark, the extra fraction of a subdivision must be judged and estimated by the eye.

Let us suppose the inches are divided into tenths. Let one point of the compasses be at O, and let the other fall at A. Now A is half way between 1.3 and 1.4; hence, as far as can be judged by the eye, the distance OA is $1.35''$. In the same way the point B is about a third of a subdivision farther than 2.2, hence OB is about $2.23''$. Similarly, OC is about $2.76''$.

In this way, after some practice, the student ought to be able to read off distances correctly to the nearest hundredth of an inch.

6. In scientific measurements the centimetre is generally employed as the unit of length, and therefore it is necessary that the student should make himself familiar with its use from the very beginning.

The centimetre is one-hundredth part of a metre, which itself was intended to be one-ten-millionth part of a quadrant of the earth's meridian, reckoned from the North Pole to the Equator.

The length of a metre is 39.37079 . . . inches, *i.e.* somewhat more than a yard; hence a centimetre is somewhat less than .4″.

We have the following table:—

10 millimetres (mm.)	make	1 centimetre (cm.)
10 centimetres	,,	1 decimetre (dm.)
10 decimetres	,,	1 metre (m.)

From the above it follows that a centimetre = .3937 . . . inches, and an inch = 2.5399 centimetres; or roughly, 1 cm. = .4″, and 1″ = 2.5 cm.

EXERCISES I

1. Express 7386 mm. in metres, decimetres, etc.

2. Express 3043 mm. in centimetres.

3. Express 46.5 dm. in centimetres.

4. Write down in centimetres:—2 m. 7 dm. 3 cm., 5 m. 5 cm. 8 mm., 2 m. 8 cm. 6.5 mm., 5 cm. 5 mm., 7 mm., 4 dm. 6 mm., 2 m. 5 mm.

5. Draw a line 5 inches long, measure it in centimetres, and hence find the number of centimetres in an inch.

6. Draw a line 10 cm. long, measure it in inches, and hence find the length of a centimetre in inches

7. Draw a line 4 inches long, divide it at random into three unequal parts, measure the lengths of the parts in millimetres, and add them; hence find the number of millimetres in an inch.

8. Draw a line 76 mm. in length, and measure it in inches.

9. Take a line 3.6″ in length, and measure it in centimetres.

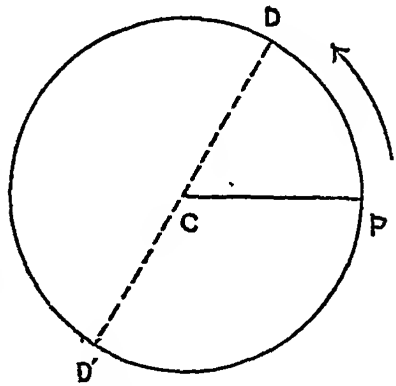
10. How many times will 25 metres of cotton go round a reel 5 centimetres in circumference?

11. The diameter of a rupee is 30 millimetres; how many of them placed in a continuous straight line would reach 3 kilometres? (1 kilometre = 1000 metres.)

12. The height of the barometric column is 76.6 cm. Express it in inches.

The Circle

7. Fix the needle point of the compasses at C, and let the pencil point P travel round and trace out a figure, taking care that the distance between the two points remains unchanged. This figure is called a **circle**. The point C is called the **centre**, and the path of P is called the **circumference**. A straight line drawn from the centre to the circumference is called a **radius**.

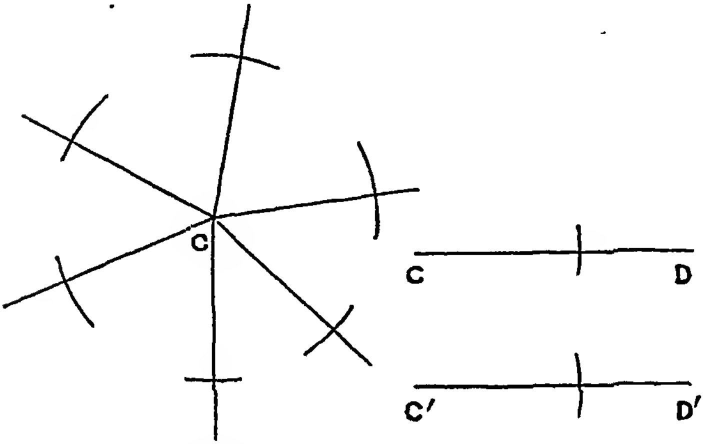


Any straight line, as DD', which passes through the centre, and is terminated both ways by the circumference, is called a **diameter**. From the manner of its construction it is evident that *in a circle all straight lines drawn from the centre to the circumference are equal;* i.e. *all the radii of a circle are equal.*

It also follows that *a diameter is bisected, or divided into two equal parts, at the centre.*

Any part of the circumference of a circle, as PD, is called an **arc**. An arc DPD' cut off by a diameter is called a **semicircle**.

8. Through the point C draw a number of straight lines in different directions. With centre C and radius $\frac{3}{4}''$ describe a circle. Then it will cut off from all these lines



lengths equal to $\frac{3}{4}''$. Thus the circle may be used to mark off equal distances on given lines radiating from the same point.

When the lines are drawn from different points, as in the second figure, we have only to keep the radius constant and change the centre; the arcs will mark off distances of $\frac{3}{4}''$ on the lines CD, $C'D'$.

EXERCISES

1. Draw a straight line 2″ long, and with its extremities as centres draw two circles of radii 2″ each.

2 Draw the straight line AB two inches long; with centres A and B, and radii of 2″, describe circles cutting one another in the points C and D; with centres C and D, and radii CA, DA, describe two other circles

3. Describe three circles with radii 3 cm., 5 cm., and 7 cm., and all having the same centre.

(Circles having the same centre are called *concentric circles.*)

4. Draw four concentric circles with radii 1, 2, 3, and 4 inches.

5. Draw two circles with radii 2″ and 3″ and centres 1.5″ apart.

6. Draw two circles with radii 2″ and 3″ and centres 5″ apart.

7. Draw two circles with radii $1\frac{1}{2}$″ and $2\frac{1}{2}$″ and centres 1″ apart.

8. Draw the line AB three inches long; cut off AC one and a half inch; with centre C describe a semicircle on AB as diameter.

9. Describe a semicircle of 6 cm. diameter.

10. Draw a straight line 5″ long, and with its extremities as centres describe circles of radii 1.5″ and 3.5″.

11. Draw a straight line 2″ long, and with its extremities as centres describe circles of radii 1.5″ and 3.5″.

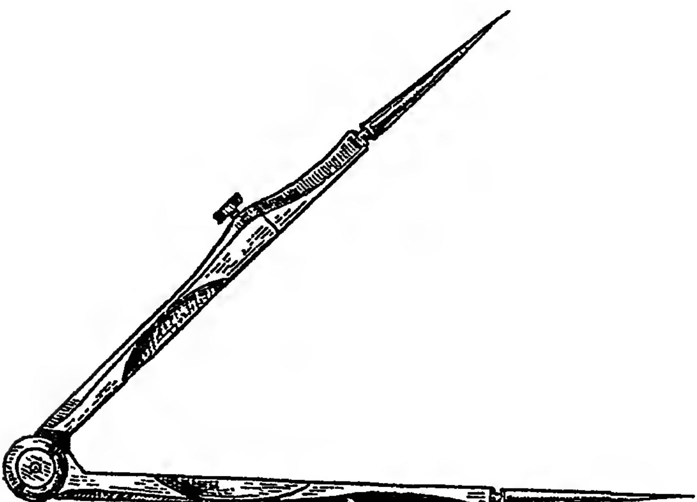
12. Draw a straight line, and from it cut off $AB=BC=CD=DE$; on AE as diameter describe a semicircle lying above AE, and on BC and CE as diameters describe semicircles lying below AE.

Draw two figures taking AB equal to 1.5″ and 3 cm. respectively.

13. In the last example take AB equal to 1″, and with B, C, D, and E as centres describe circles of radii 1, 2, 3, and 4 inches respectively.

Angles

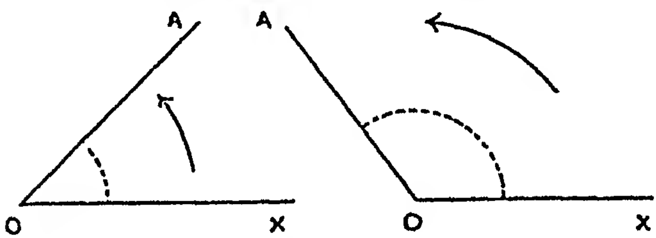
9. Lay your compasses flat on the table. With the fingers of the left hand press down and keep fixed one of



the legs, and with the right hand open them out by turning the other leg round the joint.

The opening between the legs is called an **angle.** It is evident that the angle will be greater or less according as the amount of turning is greater or less; hence the angle between the two legs is measured by the amount of turning given to the moving leg to bring it into a particular position.

Let the line *OA* originally coincide with the line *OX*. Turn round *OA* till it comes into the position shown in



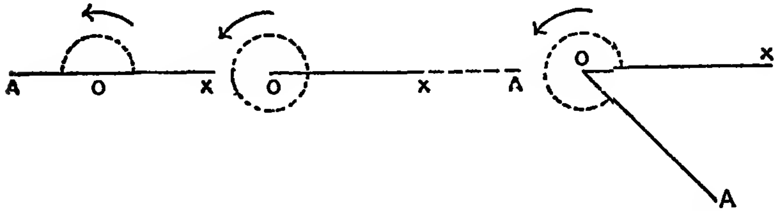
the figure. Then *OA* makes an angle with *OX*. The point *O* is called the **vertex** of the angle, and the containing lines *OX*, *OA* its **arms.** An angle is named by three letters, the letter at the vertex being put in the middle. Thus the angle in the figure is called the angle *XOA* or *AOX*.

When there is only one angle at a point, it may be denoted by a single letter, as the angle at *O*.

From the mode in which the angle *XOA* is generated, it is evident that its magnitude does not depend upon the lengths of the lines *OX*, *OA*, by which it is contained.

10. In turning round the point *O*, the line *OA* may come into a position exactly opposite to *OX*, or it may take a complete turn and again coincide with *OX*. In an

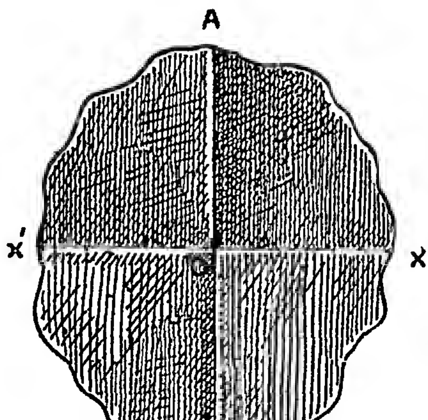
intermediate position, as in the third figure, the angle is called a **reflex angle**, and must be shown, by means of



a dotted arc, to distinguish it from the smaller angle contained by the same lines.

Right Angle

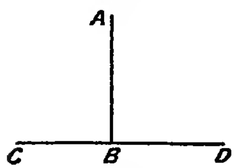
11. Take a piece of paper and fold it once; then fold it again, so that the first crease is doubled exactly on itself. Now unfold and lay the paper flat on the table.



The creases crossing at the point *O* make four angles. From the mode of their formation, it is evident that the

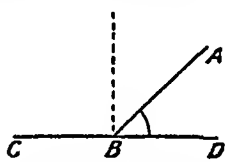
four angles are equal to one another. Each of these angles is called a **right angle.**

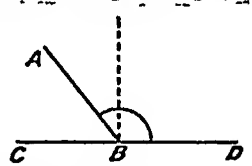
We notice that the line OA makes with the line XX' two equal angles, AOX, AOX', situated on either side of itself. Hence *when a straight line standing on another makes the adjacent angles equal, each of these angles is called a right angle; and the straight lines are said to be at right angles to each other.*



Thus if the angles ABC and ABD be equal to each other, then the lines AB and CD are said to be at right angles. The line AB is called **the perpendicular** from A on CD, and the two lines AB and CD are also said to be **perpendicular** to each other.

An **acute angle** *is less than a right angle, and an*





obtuse angle *is greater than a right angle.*

12. In the first figure of the last article the angular space about the point O is exactly filled up by four right angles.

Hence the most obvious unit of angular measurement is a right angle, and all other angles may be expressed as multiples or submultiples of a right angle. However, in order to avoid fractions, a right angle is divided into ninety equal parts called **degrees**; the degree is subdivided into sixty equal parts called **minutes**, and the minute into sixty equal parts called **seconds**. Thus $\frac{3}{16}$ of a right angle is equal to *sixteen degrees, fifty-two minutes, and thirty seconds*; and is written $16° 52' 30''$.

EXERCISES II

1. What is the number of degrees in half a right angle, in one-third of a right angle, and in two-thirds of a right angle respectively?

2. In the first figure of Art. 11, how many degrees are there in the angle between OX and OX'? and how many in the reflex angle XOA'?

3. Convert $\frac{25}{32}$ of a right angle into degrees, minutes, and seconds.

4. How many seconds are there in $\frac{11}{48}$ of a right angle?

5. What fraction of a right angle is $30° \, 22' \, 45''$?

6. Convert $43° \, 8' \, 15''$ into the fraction of a right angle.

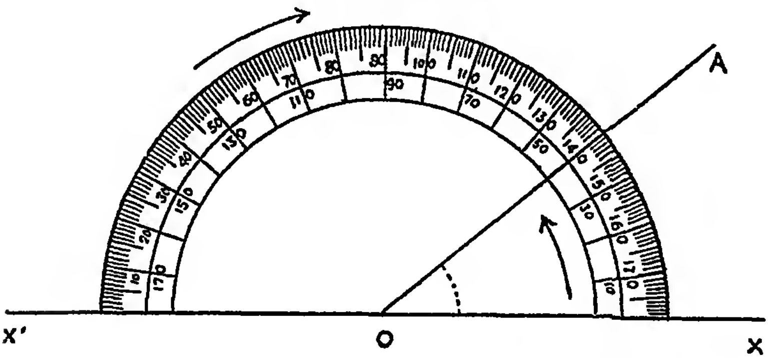
7. How many degrees does the long hand of a clock turn through in an hour? and how many in 42 minutes?

8. Express in right angles the sum of the three angles $29°, 38' \, 11''$, $89° \, 0' \, 25''$, and $61° \, 21' \, 24''$.

9. A straight line is turned through an angle of $135°$, and again through an angle of $187°$; through how many degrees should it be turned farther, so as to bring it to its original position?

10. Find the number of degrees between the hands of a clock at a quarter past 12.

13. Measurement of Angles.—The protractor is used for measuring angles. It has a base and a semicircular rim marked with two rows of figures. The double row enables



you to measure angles either to the left or right, by reckoning from the base line in each case.

Suppose it is required to measure the angle XOA. Place the protractor so that its centre coincides with the vertex O of the angle, and its base line coincides with the arm OX. If the other arm OA passes through the sub-

division marked 40 on the inner row, the angle XOA is 40°.

If XO be produced to X', then the angle $X'OA$ can be read off on the outer row of figures, and will be found to be 140°. We might have expected this, for the sum of the two angles is the angle between OX and OX', which is 2 right angles, or 180° (Art. 11).

On the other hand, suppose we are required to draw a line OA, making an angle of 40° with OX. Place the protractor as before. Seek out with your pencil the point exactly opposite to the subdivision marked 40 on the rim. The line drawn from this point to O will make an angle of 40° with OX.

EXERCISES III

1. Make angles of various sizes, and measure them, and in each case test your result by measuring the corresponding angle which one of the arms produced makes with the other arm.

2. Use the protractor to construct angles of 15°, 30°, 45°, 60°, 72°, 18°, 36°, 54°, 90°, 120°, 150°, 135°, 210°, 216°, 270°, 315°, and 330°.

3. Take a straight line AB two inches long. At A and B make angles of 30° and 60° respectively, on the same side of AB. Let the containing lines meet in C. Measure the angle at C, and also the length of the line BC.

4. Take a straight line 6 cm. long, and perform the same operations as in the last example.

5. Draw the lines CX, CY containing an angle of 72°. Cut off CA, CB, each equal to 2 inches. Join AB. Measure the angles CAB, CBA. If your drawing is correct, each angle ought to be 54°.

6. Draw the lines CA, CB, each 5 cm. long, and containing an angle of 60°. Measure the length of AB and the angles CAB, CBA.

7. Measure AB three inches long. At A and B make angles of 60°, and let the containing lines meet in C. Measure the angle at C, and also the lengths AC and BC.

8. Describe a circle of 2″ radius, and draw radii at angular distances of 60° from each other. Join the extremities of these radii, and measure the sides and angles of the six-sided figure so formed.

9. Take a line AB 10 cm. long. At A and B make the angles CAB, CBA equal to 40° and 55° respectively. Find by measurement the lengths of AC and BC to the nearest millimetre; also measure the angle ACB.

10. Make the angle XAB of 60°, and on the other side of the line AB make the angle $X'AB$ of 120°.

Is XAX' a straight line?

11. Draw a figure placing the two angles 30° and 150° adjacent to each other.

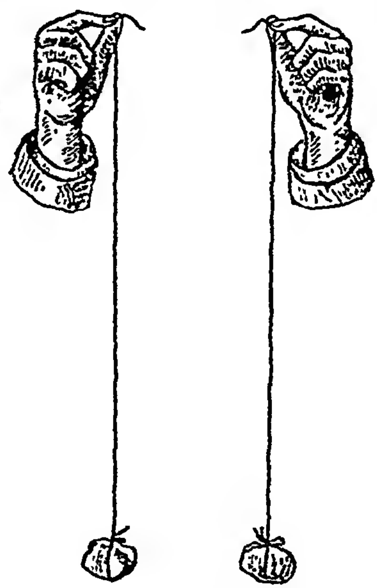
12. Draw a figure placing the angles 40°, 60°, and 80° adjacent to each other.

13. Draw a circle of 2" radius; rule the diameter AC; take B and D any two points on the circumference and join AB, BC, CD, and DA.

Measure the angles at B and D, and find the sum of the angles BAD, BCD.

14. Make the angle AOB of 35°. Produce BO to B', and verify by measurement that the sum of the angles AOB, AOB' is 180°.

15. Draw a circle of 5 cm. radius, and take any four points on the circumference. Join the points so as to form a four-sided figure. Verify by measurement that the sum of the opposite angles of this figure is equal to two right angles.



Parallel Straight Lines

14. Suspend two weights by two fine threads, so as to hang not far from each other. These threads will represent **parallel straight lines.**

The first thing we notice about them is that they are at the same distance from each other throughout their lengths.

It is obvious that we can make the suspending threads

as long as we like and yet keep them at the same distance from each other.

Thus *parallel straight lines are such as remain at the same distance from each other.*

It also follows that *if parallel straight lines are continually produced ever so far both ways they cannot meet each other.*

EXERCISES IV

1. Let the student point out all the lines which appear to him to be parallel: (i.) on a rectangular block of wood; (ii.) in the room in which he is sitting; (iii.) on a table; (iv.) on the cover of a book; and (v.) on a sheet of paper.

2. Take a straight line, and with the help of your protractor, through different points on it draw straight lines, making a right angle with it. Do these lines appear to be parallel?

3. Take a straight line, and through different points in it draw straight lines, all making the same angle (say an angle of 60°) with it in the same sense.) Do these lines appear to be parallel?

4. Repeat the last exercise for angles of 30°, 36°, 45° 60°, 75°, 120°, and 135°.

The last three exercises exhibit parallel lines in another aspect, viz. *parallel straight lines are such as make the same angle with a given straight line*, or, in other words, *parallel straight lines are drawn in the same direction.*

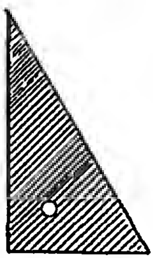
CHAPTER II

SIMPLE CONSTRUCTIONS

15. THE student has already learnt the use of the straight edge, compasses, and protractor. We shall now see how (the set-squares may be used in drawing perpendiculars and parallels, and in making the figures of a rectangle and square.)

The Set-Squares

16. Each set-square has the shape of a triangle, *i.e.* a figure contained by three straight lines. One of the angles of this triangle is a right angle, and the other two are acute. The acute angles in (1) are angles of 60° and 30° respectively, and in (2) the acute angles are each of 45°. Just as the straight edge is used in drawing straight lines, a set-square may be used in making a right angle by drawing the pencil along the edges which contain the right angle. Thus our right angle will be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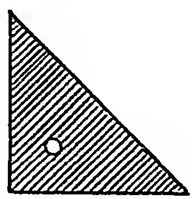
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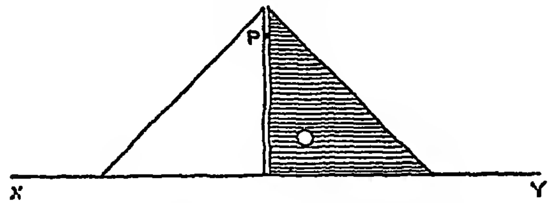
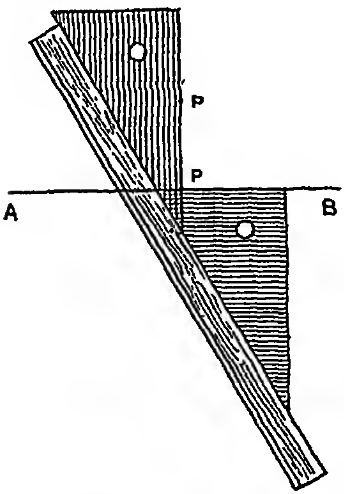
accurate or otherwise according as the set-square is accurately right-angled or not.

In order to test your set-square, rule a line XY, and



take a point P outside it. Place one of the edges containing the right angle along the line XY, and make the other edge pass through P; rule a line through P along this edge.

Turn the set-square over into the shaded position in the figure and again rule a line through P. If the set-square is accurately right-angled, these two lines must be the same.



Perpendiculars

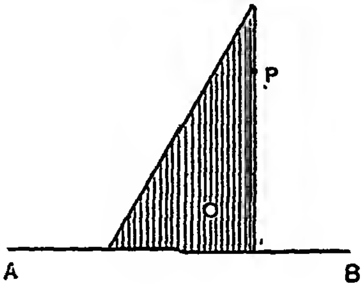
17. *To draw a perpendicular to a given line AB from a point P on it or outside it.*

Place a short edge of the set-square in contact with AB, and bring the straight-edge, or another set-square, in contact with the longest edge.

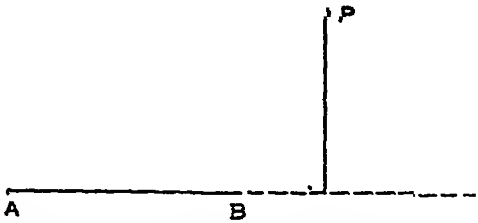
Now slide the set-square until its other edge passes

through P, which may be on the line or outside it. Rule a line through P. This will be the required perpendicular.

This is the proper method. For if we place the set-square as in the second figure, the line through P will have a tendency to slip towards A in the vicinity of the right angle, and it may be otherwise imperfect on account of the wearing away of the corner.



The student ought to practise drawing perpendiculars by taking the point P in a variety of positions. He will find that sometimes



the perpendicular through P will not meet AB, but it will meet AB produced as in the figure.

EXERCISES V

1. Take AB one inch long, and from the extremity B draw BC at right angles to AB, and cut it off one inch long. Join AC. Measure the angles at C and A, and find the length of AC correct to the nearest tenth of an inch.

2. Take a line AB of 6 cm. Through its middle point M, which can be found by measuring, draw MC at right angles to AB and 3 cm. in length. Join AC, BC, and measure the angle ACB. Also measure the lengths of AC and BC correct to the nearest millimetre.

3. Draw XY 10 cm. Erect the perpendicular YZ, and make it equal to 17.3 cm. Join ZX. Measure the length of ZX to the nearest millimetre, and the angles at Z and X to the nearest degree.

4. Describe a circle of 1″ radius and rule a diameter. Draw another diameter at right angles to the first, and join the extremities of the diameters so as to form a four-sided figure. Measure the sides and angles of this figure.

Repeat this exercise with a circle of 3 cm. radius.

5. Take AB 2″ long. Through B draw a perpendicular to AB. With centre A and radius 4″ describe an arc of a circle cutting the perpendicular in C. Join AC. Measure the angles at A and C, and estimate the length of BC correct to the nearest hundredth of an inch.

6. Take any three points and join them so as to form a triangle. Draw perpendiculars from the angular points of this triangle on the opposite sides. These perpendiculars ought to pass through the same point.

Repeat this exercise for a variety of cases.

7. Describe a circle 3 cm. in radius. Take any three points on the circumference and join them by straight lines forming a triangle. Now take any other point on the circumference, and through it draw perpendiculars to the sides. If your drawing is correctly made, the feet of these perpendiculars will lie in a straight line.

Repeat this exercise for circles of different radii.

8. Describe a circle with centre C and of 1″ radius. Draw a straight line cutting the circle in the points A and B, but not passing through the centre. Through C draw CO perpendicular to AB. Are OA and OB equal to each other?

9. Draw AB two inches long, and draw the perpendiculars BC, AD on the same side of AB and equal to it in length. Join CD, and measure its length.

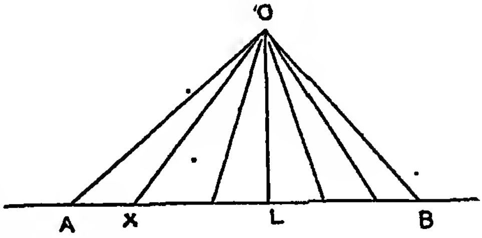
Also join AC, BD cutting one another in O, and with centre O and radius OA describe a circle which ought to pass through B, C, D.

10. Draw AB three inches long, and draw the perpendiculars BC, AD on opposite sides of AB and equal to it in length. Join CD cutting AB in O, and with centre O and radii OA, OC describe two concentric circles which ought to pass through B and D respectively.

11. Repeat the last two exercises taking AB seven centimetres in length.

12 Draw AB four inches long, and through A draw four lines making angles of 30° and 60° with AB and lying both above and below AB. Through B draw four perpendiculars to these lines, and with AB as diameter describe a circle which ought to pass through the feet of the perpendiculars.

18. Take a line AB and a point O lying outside it. Draw OL perpendicular to AB, and through O draw other lines to the right and left of OL and meeting AB. The student can easily test, by



measuring, that OL is the shortest line that can be drawn from O to meet AB.

Hence *the shortest distance of a point from a straight line is the length of the perpendicular drawn from the point to the straight line.*

The student ought to bear in mind that when we speak of the distance of a point from a straight line we always mean the perpendicular distance.

EXERCISES VI

1. Make an exact copy of your 45° set-square, and measure the distance of the right angle from the longest side. Is it exactly equal to half the longest side?

2. Take AB four inches long, and through B draw a line making an angle of 30° with AB. Measure the distance of A from this line. Is it exactly equal to one half of AB?

3. Take any line AB. Through B draw the straight line BC making an angle of 150° with AB, and of length equal to 2″. Find the distance of C from AB, and measure the angle which this distance makes with BC.

4. Take the straight line AB of any convenient length. Draw the perpendiculars AL and BM each of length 1″. Join LM. Find the distances from AB of different points on the line LM. Are they all equal?

5. Measure AB two inches in length. At A and B erect perpendiculars AD and BC, each two inches in length. Join CD, and also AC and BD, cutting one another in O. Measure the distances of O from the sides of the figure $ABCD$. Are they all equal?

6. Measure *AB* five inches long, and draw *BC* perpendicular to *AB* and equal to it in length. Join *AC*, and show from your figure that the length of *AC* is double the distance of *AC* from the point *B*.

7. Take a line *AB* and a point *O* outside it; draw a series of circles with *O* as centre and cutting the line *AB* in two points, which will come closer to one another as the radius is decreased, until you arrive at a circle which meets the line *AB* in one point *L* only. Show that *OL* is perpendicular to *AB*.

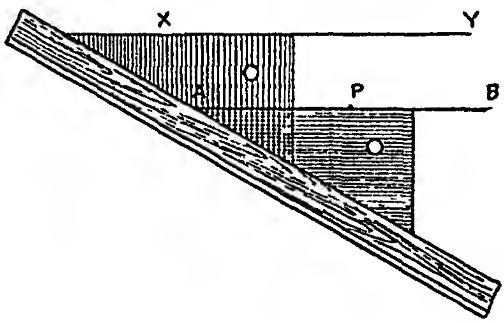
8. Draw *AB* ten centimetres in length, and make the angle *BAX* of 60°. Show that the distance of the point *B* from the line *AX* is 86 millimetres nearly.

Parallels

19. *Through a given point P to draw a straight line parallel to a given straight line XY.*

Place a set-square so that a short edge coincides with *XY*.

Bring the ruler in contact with the longest edge, and

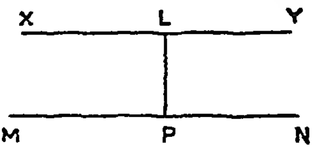


keep it firmly fixed. Now slide the set-square along the ruler until the edge which was originally in contact with *XY* passes through *P*. Rule a line *AB* through *P* along this edge. Then *AB* will be parallel to *XY*.

The student will notice that in the figure both *XY* and

AB make the same angle of 30° with the straight-edge (Art. 14).

20. *To draw a straight line parallel to a given straight line XY and at a given distance from it.*



Take any point L in XY. Draw LP at right angles to XY, and cut off LP equal to the given distance. Through P draw a straight line MN parallel to XY, as in the last article.

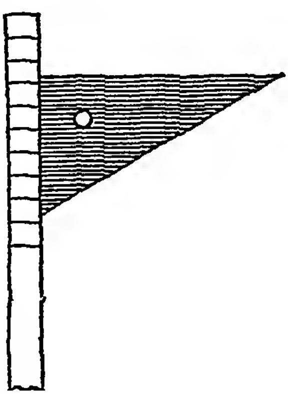
Draw a line XY, and another line MN parallel to it and at a distance of $\frac{3}{4}''$ from it. It will be found on measurement that the distances of all points on either line from the other is the same, and equal to $\frac{3}{4}''$. Moreover, it will be found that the common perpendicular is the shortest distance from one line to the other.

EXERCISES VII

1. Draw a set of four parallel lines, and draw another set of four parallel lines cutting them.

2. Draw five parallel lines, and another four parallel lines cutting the former at right angles.

3. Draw a set of five parallel lines at distances of 1″, using the graduated ruler and set-square as shown in the figure. Then draw another set of parallels at distances of 1″. We thus obtain a number of four-sided figures. Measure their sides and angles. Are they all equal?



4. Draw six parallels at distances of 1 cm., and then draw another six parallels at right angles to them and at distances of 1 cm.

5. Take any triangle, and through its angular points draw parallels to the opposite sides. You will thus obtain a larger triangle; measure the sides and angles of this triangle, and compare them with those of the original.

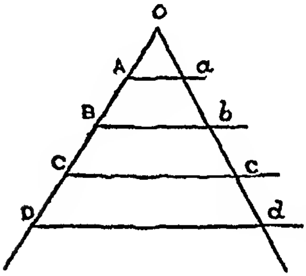
6 Take any line, and draw a set of seven lines cutting it at intervals of half an inch and making angles of 30° with it.

7. Take a line 5″ long and draw six lines, each 5″ long, cutting the first line at intervals of an inch and making angles of 60° with it.

8. Measure AB five inches in length, and draw BC, AD perpendiculars to AB, lying on the same side of AB and equal to it in length. Join CD, and draw parallels to AB, BC at intervals of an inch. You will find that the figure $ABCD$ is divided into 25 equal figures of the same shape as $ABCD$.

Division of a Line into Equal Parts

21. Draw a line through the point O, and step off any number of equal distances OA, AB, BC, etc., on it. Through A, B, C . . . draw a set of parallels. Then if any other line be drawn through O to cut these parallels, its parts Oa, ab, bc . . . will all be found on measurement equal to one another.



As an example of the above, rule a line OX and step off the distances OA, AB, BC . . ., each equal to 1″. Through A, B, C . . . draw parallels at right angles to OX, and draw the line Ox, making an angle of 60° with OX, and cutting the parallels in a, b, c . . . On measuring, you will find that each of the parts Oa, ab, bc . . . is equal to 2″.

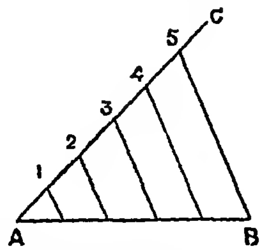
Thus *parallel lines which are drawn at equal distances from each other cut off on any other line equal intercepts.*

This property of parallel lines is utilised in making equal divisions of a straight line.

22. *To divide a given straight line AB into any number of equal parts (say five).*

Draw AC, making any angle with AB. Take any

distance A 1 and lay it off five times along the line AC, so that A 1, 1 2, 2 3, 3 4, and 4 5 are all equal to each other. Join 5B.



Now place one of the short edges of the set-square to coincide with 5B, and bring the straight-edge in contact with the other short edge. Rule parallels to 5B through 4, 3, 2, and 1. These parallels will divide AB into five equal parts.

The construction given above enables us to divide a given line AB into parts having a given ratio. For suppose the given ratio is that of 3 : 5. Divide AB into 8 equal parts, and let AC contain 3 of these parts; then AB is divided at the point C as required.

EXERCISES VIII

1. Divide a line 2″ long into three equal parts, and find the length of each part by measurement.

2. Divide a line 5″ long into seven equal parts, and measure the length of each part as near to the hundredth of an inch as you can.

3. Bisect a line 5 cm. in length; *i.e* divide it into two equal parts.

4. Draw any triangle, bisect two of its sides, and draw the line joining the points of bisection. Measure the length of this line, and compare it with the length of the third side.

5. Divide a line 7.8″ long into six equal parts, and also into two parts having the ratio of 5 : 8.

6. Divide a line 4.2 cm. long into three parts which are in the ratio of 1 : 4 : 9.

7. Describe a circle, centre O and radius OA equal to 4 cm.; bisect OA in L, and draw LT at right angles to OA to meet the circumference in T. Measure the angle TOL, and find by measurement the length of TL correct to the nearest millimetre.

8. Draw any triangle; bisect each of the three sides, and through the points of bisection draw perpendiculars to the sides. Do these perpendiculars meet in a point? Are the distances of this point from the angular points all equal?

9. Divide a line 3″ long into four equal parts, and also into five equal parts. Measure the difference between two parts correct to the hundredth of an inch.

10. Divide a line 5.4″ into two parts so that one part is double of the other.

11. Divide a line 52 millimetres in length into two parts so that one part is three times the other.

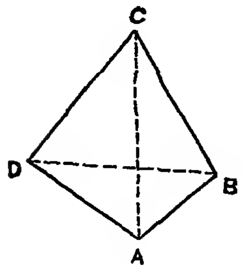
12. Divide a straight line 5″ long in the proportion of 2 : 3 : 5.

13. Draw a line 5.1″ long and trisect it.

14. Draw any triangle and trisect each of its sides; join the points of division by three sets of lines which are parallel to the sides of the triangle. The triangle will be divided into nine equal triangles.

15. Repeat the last exercise dividing each side of the triangle into four equal parts.

Four-sided Figures

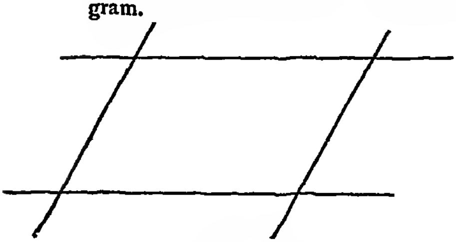


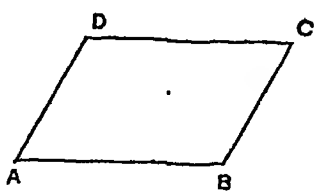
23. Any figure contained by four straight lines is called a **quadrilateral**.

The lines (*AC* and *BD*) joining the opposite angles of a quadrilateral are called its **diagonals**.

A **parallelogram** is a quadrilateral whose opposite sides are parallel.

Draw a pair of parallels, and another pair of parallels cutting them both; we thus obtain a four-sided figure, which is a parallelogram.





Again, take *AB* 3 inches in length; make the angle *BAD* of 60°, and cut off *AD* equal to 2 inches. Draw *DC*, *BC* parallel to

AB and *AD* respectively; then *ABCD* is a parallelogram having sides 3″ and 2″, and the contained angle of 60°.

EXERCISES IX

1. Draw any quadrilateral, and measure all its angles. Is their sum equal to four right angles?

Repeat this exercise with a number of quadrilaterals drawn at random.

2. Construct a parallelogram whose adjacent sides are 5 cm. and 3 cm. and the contained angle of 50°. Measure its other sides and angles, and make a note of anything you discover.

3. The sides of a parallelogram are 3″ and 4.2″, and the contained angle is of 150°. Construct it, and measure its other sides and angles.

4. Make a parallelogram of sides 7 cm. and 5 cm. and contained angle of 72°. Draw the diagonals and see if they bisect each other.

5. Construct the following parallelograms, measure their sides and angles, and test in each case that the diagonals bisect each other:

(i.) Sides 3.5″ and 2.2″; included angle 30°. (ii.) Sides 4 cm. and 2.5 cm.; and included angle 108°. (iii.) Sides 1.5″ and 2.6″; and included angle 45°. (iv.) Sides 1.75″ and 2.3″; included angle 54°. (v.) Sides 2.7 cm. and 1.8 cm.; included angle 36°.

6. *A rhombus is a parallelogram having adjacent sides equal.* Construct rhombi, having the following sides and angles; measure all the sides, and verify that the diagonals bisect each other at right angles:

(i.) Side 3″, angle 60°. (ii.) Side 4.5 cm., angle 105°. (iii.) Side 2.6″, angle 48°. (iv.) Side 3.3 cm., angle 150°. (v.) Side 2.8″, angle 120°.

7. Draw *AB* 4 inches long, make the angle *BAD* of 60°, and cut off *AD* equal to 2 inches. Complete the parallelogram *ABCD*, draw the diagonal *BD*, and measure the angle *ADB*.

8. Construct a parallelogram, sides 4″ and 5″ and included angle 67°. Divide the sides into inches (Ex. 3, Exer. VII.), and through the points of division draw parallels to the sides. The parallelogram is thus divided into twenty equal parallelograms.

9. Take any quadrilateral and join the middle points of the sides. Is the four-sided figure thus formed a parallelogram? Measure the sides of this figure, and discover any relation they may bear to the diagonals of the quadrilateral.

10. Draw *AC* ten centimetres in length and find by measurement its

middle point O; through O draw BOD at right angles to AC and cut off $OB = OD = 3$ cm. Show that the figure $ABCD$ is a rhombus.

11. Repeat the last exercise taking AC six inches and OB four inches in length, and show that each side of the rhombus is 5 inches in length.

12. Construct a rhombus whose diagonals are 6 and 8 centimetres in length respectively.

13. Make a parallelogram whose sides are 3.5 and 7 cm. in length, and the contained angle is of 60°.

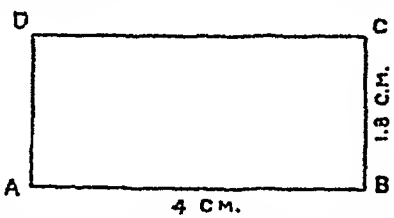
14. Through the point O draw two lines XOX', YOY' containing an angle of 30°; in the line XX' cut off $OA = OC = 3''$, and in the line YY' cut off $OB = OD = 4''$; join AB, BC, CD, and DA. Verify that the figure $ABCD$ is a parallelogram.

15. In the last exercise if the lines XX', YY' intersect one another at right angles, show that the figure $ABCD$ is a rhombus of side 5".

24. From the exercises of the last article the student must have discovered that *the opposite sides and angles of a parallelogram are equal, and that its diagonals bisect each other; and that in the case of a rhombus, all the four sides are equal, and the diagonals bisect each other at right angles.*

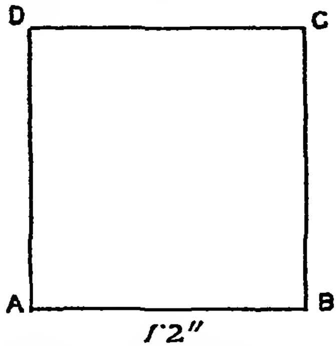
We shall now consider two special kinds of parallelograms, viz. the rectangle and the square, which are of importance in the measurement of areas.

A **rectangle** is a parallelogram with one angle a right angle; and a **square** is a rectangle with equal sides.



Draw AB 4 cm. in length, draw AD at right angles to AB, and cut it off 1.8 cm. in length. Through B and D draw BC, DC parallel to AD and AB. Then $ABCD$ is a rectangle of sides 4 cm. and 1.8 cm.

Again, take AB 1.2″ in length. Draw the perpendicular AD,



and cut it off equal to AB. Draw BC, DC parallels as before; then $ABCD$ is a square of side 1.2″.

EXERCISES X

1. Construct the rectangles whose sides are given below; measure their sides, angles, and diagonals. Note any peculiarity that you had not noticed in the ordinary parallelogram:

(i.) 3″ and 2″. (ii.) 2.4 cm. and 4.7 cm. (iii.) 3.6″ and 2.5″. (iv.) 4.8″ and 3″. (v.) 6.3 cm. and 8.8 cm.

2. Construct the squares whose sides are given; measure the sides and diagonals, and also measure the angles which the diagonals make with the sides:

(1) 3.2″. (2) 7.2 cm. (3) 2.3″. (4) 6 cm. (5) 1″. (6) 1 cm.

3. Construct a rectangle of sides 4.5″ and 2.8″; join the middle points of its sides, so as to form a four-sided figure. Measure the sides of this figure, and find out what it is.

4. Describe a circle of 1.4″ radius. Draw two diameters, and join their extremities. Measure the angles of this figure, and say what it is.

5. Describe a circle of 3 cm. radius. Draw two diameters at right angles, and join their extremities. What is the figure you obtain?

6. Repeat the last two exercises with circles of different radii.

7. Describe a circle of 2″ radius. Draw the diameter AC. Take B any point on the circumference, and join BA, BC. Through A and C draw parallels to BC and BA. Do these parallels intersect on the circumference of the circle? and what sort of a parallelogram have you obtained?

8. The diagonal of a square is 3″. Construct the square by using one of your set-squares only.

9. Construct a rectangle of sides 5″ and 3″. Divide the sides into inches, and through the points of division draw parallels to them (see Ex. VII., 3). Is the rectangle divided into 15 squares of 1″ side?

10. Make a rectangle with sides 4 cm. and 7 cm. By a construction similar to that of the last exercise divide the rectangle into 28 squares of 1 cm. side.

11. Through a point *O* draw two straight lines at right angles; with centre *O* and a radius of 10 cm. describe a circle cutting the lines through *O* in *A*, *B*, *C*, and *D*. Join *AB*, *BC*, *CD*, *DA*, and show that *ABCD* is a square each of whose sides is about 141 millimetres in length.

12. Take *AB* seven centimetres in length; through *A* and *B* draw four lines, making angles of 45° with *AB*. These lines will form a square each of whose sides is about 5 centimetres.

13. Through a point *O* draw two straight lines containing an angle of 60°; with *O* as centre and a radius of 2·7″ describe a circle cutting the lines through *O* in *A*, *B*, *C*, *D*. Show that the figure *ABCD* is a rectangle one of whose sides is 2·7″.

14. Describe two squares whose sides are 1″ and 2″ respectively; show how to divide the larger square into four squares each of which is equal to the smaller square.

15. Describe two squares whose sides are 1.4″ and 4.2″ respectively. Trisect two adjacent sides of the larger square and through the points of division draw parallels to the sides. Show that the larger square contains nine of the smaller square.

16. Make a rectangle of sides 3″ and 4″, and show after the manner of Ex. 9 that it contains 12 squares of 1″ side.

17. Construct a rectangle of sides 4 cm. and 5 cm. and show by means of a diagram that it contains 20 squares of 1 cm. side.

18. The sides of a rectangle are 3·5″ and 4″; prove by means of a figure that it contains 14 complete squares of 1″ side.

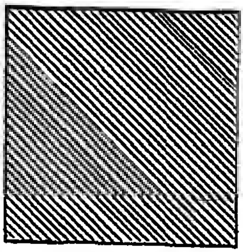
19. Construct a square *ABCD* of 3″ side; join *AC*, *BD*; through *A*, *C* draw parallels to *BD*, and through *B*, *D* draw parallels to *AC*. Verify that the figure enclosed by these parallels is a square, and show, by counting triangles, that the area of this square is twice the area of the square *ABCD*.

CHAPTER III

MEASURES OF LENGTHS AND AREAS

25. **Units of Length and Area.**—Every magnitude is measured by considering how many times a selected standard, or "unit" of the same kind as itself, is contained in it. Thus in measuring length we may take for our unit an inch, a foot, or a yard; and then the length of any line may be measured by finding how many times one of these units is contained in it. When the unit of length has been selected, the unit of area is derived from it. For the unit of area is defined to be the area of a square, each of whose sides is the unit of length. For example, if a foot be taken as the unit of length, the square whose side is 1 ft. is the unit of area; and any other area will be measured by finding how many times this square is contained in it.

Similarly, if an inch is taken as the unit of length, the square whose side is 1 inch is the unit of area.



Square Inch.

26. We give here for reference two tables containing the measures of length and area.

MEASURES OF LENGTH

12	Inches (12 in.)	make	1 Foot (1 ft.).
3	Feet	,,	1 Yard (1 yd.).
$5\frac{1}{2}$	Yards	,,	1 Pole (1 po.).
40	Poles (220 yds.)	,,	1 Furlong (1 fur.).
8	Furlongs (1760 yds.)	,,	1 Mile (1 mi.).

MEASURES OF AREA

144	Square Inches (sq. in.)	make	1 Square Foot (1 sq. ft.).
9	Square Feet	,,	1 Square Yard (1 sq. yd.).
$30\frac{1}{4}$	Square Yards	,,	1 Square Pole (1 sq. po.).
40	Square Poles	,,	1 Rood (1 ro.).
4	Roods (4840 sq. yds.)	,,	1 Acre (1 a.).
640	Acres	,,	1 Square Mile (1 sq. mi.).

27. Land is usually measured with a Chain, invented by Mr. Gunter, and known as "Gunter's Chain." It is 22 yds. long, and is divided into 100 equal parts, called links; each of which is therefore 7.92 in.

A square chain contains 22×22, or 484 sq. yds.; hence 10 square chains make an acre.

LAND MEASURE (LENGTHS)

100 Links (22 yds.)	make	1 Chain.
1000 Links (10 chains)	,,	1 Furlong.
8000 Links (80 chains)	,,	1 Mile.

LAND MEASURE (AREAS)

10,000 Square Links	make	1 Square Chain.
100,000 Square Links	,,	1 Acre.

28. A figure drawn on paper to represent a field is of the same shape as the field itself, but its size is necessarily much smaller.

In fact we construct on paper a figure *similar* to the field by making the angles of the drawing equal to the angles of the field, and every line in the drawing the same fraction of the corresponding line in the field.

29. Such a representation of a field, or of any other surface, is said to be **drawn to a scale**, and the common ratio of all lines on the drawing to the corresponding lines on the original is called the **representative fraction** of the scale.

Thus if we agree to represent a length of a hundred feet on the field by a length of one inch on paper, the figure is said to be drawn to the scale of 1″ to 100′, and the representative fraction of the scale is $\frac{1}{1200}$. In this scale a field whose sides are 300′, 400′, and 500′ will be represented by a triangle whose sides are 3″, 4″, and 5″ respectively. Here every line in the drawing is a twelve-hundredth part of the line on the field which it represents.

30. *A* **scale** *is a divided line of convenient length used for the purpose of measuring lengths on a drawing.*

In a **plain scale** a line of known length is divided into a number of equal parts, each of which is assumed to represent a certain length.

Thus take a line six inches long and divide it into six equal parts. Then if we assume each of these parts to represent a yard, the whole line will represent six yards; and if we divide one of the parts into three equal divisions, each of these will represent a foot.

Such a scale is called a scale of one inch to the yard, or a scale of $\frac{1}{36}$; on it every division stands for a yard, every

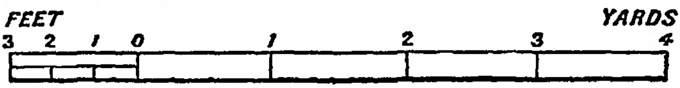
subdivision for a foot, and any length is a thirty-sixth part of the actual length it represents.

31. We shall here give two examples of the construction of plain scales. The student who has grasped the principle on which these are made will find no difficulty in working out for himself the exercises which follow. The proposed scales should be drawn neatly on narrow strips of paper and preserved for future use.

Example 1. *To draw a scale of $\frac{5}{288}$, to read yards and feet, and to make it long enough to measure 5 yds.*

Here the length of 1 yd. will be represented by $\frac{5}{288}$ of a yard, therefore the whole length of the required scale will be $\frac{5\times5}{288}$ yds., or $\frac{5\times5\times36}{288}=3\frac{1}{8}$ in.

Take a line $3\frac{1}{8}''$ long and divide it into 5 equal parts; then each of these parts will represent a yard.

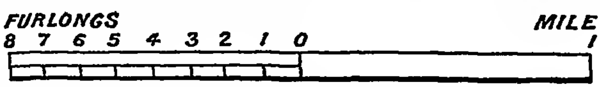


If now the first division be divided into 3 equal parts, each of them will represent a foot.

Place a 0 (zero) at the end of the first division; and mark the other divisions both ways as shown in the figure. Lastly, write the word *Feet* on the left, and *Yards* on the right. Any distance can now be taken off from the scale by the compasses. Suppose we require to take off 3 yds 2 ft.; place one point of the compasses on the mark 3 yds., and stretch out the other to the mark 2 ft., then the distance between the points will be 3 yds. 2 ft.

Example 2. *Construct a scale of $1\frac{3}{8}''$ to the mile, showing furlongs, and long enough to measure 2 miles.*

Take a line $2\frac{3}{4}''$ long and divide it into two equal parts; and



make 8 equal divisions of the first part. Each large division will stand for a mile, and each subdivision for a furlong

The representative fraction of this scale is $\frac{1\frac{3}{8}}{5280 \times 12}$, or $\frac{1}{46080}$.

Exercise 1. Draw a scale of 1 in. to the foot to read feet and inches, and make it long enough to measure 6 ft.

Exercise 2. Draw a scale of 1 in. to the chain (22 yds.), showing poles ($5\frac{1}{2}$ yds.), and long enough to measure 6 chains.

Exercise 3. Construct a scale of $1\frac{1}{4}''$ to 100′, showing 10′, and long enough to measure 400′.

Exercise 4. Draw a scale of $\frac{1}{24}$, to read feet and inches.

Exercise 5. Construct a scale of $\frac{1}{16}$. to read feet and inches, and make it long enough to measure 6 ft.

Exercise 6. The representative fraction of a scale is $\frac{1}{60}$. Draw the scale to show at least 12 yds., and to read yards and feet.

Exercise 7. The distance between Lahore and Amritsar is 32 miles, and the distance between their positions on a map is $1\frac{1}{2}$ in.; find the scale on which the map is drawn.

Ans. $\frac{1}{1351680}$.

Exercise 8. On the map of the last exercise the distance between Lahore and Jalandar is $3\frac{3}{4}$ in. What is the actual distance?

Ans. 80 miles.

CHAPTER IV

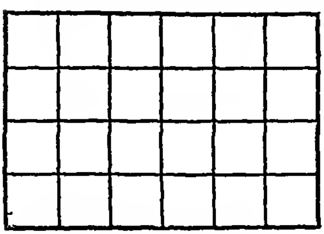
THE AREA OF A RECTANGLE

32. THE student has been shown how to construct rectangles and squares of given dimensions. We shall now investigate rules for finding their areas and apply them to the working of practical calculations.

33. *To find the area of a rectangle whose sides are of given lengths (say 6 ft. and 4 ft.).*

Construct the figure to the scale of $\frac{1}{4}''$ to the foot; and divide the side 6 ft. into 6 equal parts, and the side 4 ft. into 4 equal parts. Then each division will represent 1 ft.

Through the points of division draw lines parallel to the sides of the rectangle; these lines will divide the rectangle into a number of squares whose sides are 1 ft. in length.



Now if we take 1 ft. as the unit of length, each of these squares will represent the unit of area, since the sides of each are of unit length (Art. 25). Here we have four rows of these squares, and six in each row, therefore the whole number of squares is 6×4, or 24.

Hence

$$\text{the area of the rectangle} = 6 \times 4 \text{ units of area}$$
$$= 24.$$

Similarly *the number of units of area in any other rectangle will be found by multiplying together the numbers representing the lengths of its sides.*

34. The sides of a rectangle are usually called its **length** and **breadth**. Hence the rule for finding its area may be stated thus :—

The area of a rectangle is equal to the product of its length and breadth.

Before applying this rule we must express the length and the breadth in terms of the same unit; and then the product will represent the number of the corresponding square units. Thus if the length and breadth are both taken in feet, the area will be given in square feet; if they are both expressed in chains, the area will be found in square chains; and so on.

Ex. 1. *The length of a rectangle is* 5 *ft.* 2 *in., and the breadth is* 3 *ft.* 9 *in.; find the area.*

We have

5 ft. 2 in. = 62 in., and 3 ft. 9 in. = 45 in.;

therefore the area = $62 \times 45 = 2790$ sq. in.

Ex. 2. *The length of a rectangle is* 160 *poles, and its breadth is* 150 *links; find the area in acres.*

The length = 160 poles = 880 yds. = 40 chains = 4000 links;
and the breadth = 150 links.

Hence the area = $4000 \times 150 = 600{,}000$ sq. links
= 6 acres.

35. The length and breadth of a square are equal; hence *its area is found by multiplying the length of a side by itself, that is, by squaring the number which denotes the length of a side.*

Ex. *The side of a square is* 12 *in.; find the area.*

The area = $12 \times 12 = 144$ sq. in.

Note.—The side of this square is evidently 1 ft.; and hence

the area is 1 sq. ft. The student will now easily understand how 144 sq. in. make 1 sq. ft. (Art. 26). He ought to make a figure, as in Art. 33, and prove this statement geometrically.

36. It is well to bear in mind the distinction between *square feet* and *feet square*. Thus when we speak of 5 square feet, we mean an area which contains a square foot five times; whereas by 5 feet square we mean the area of a square whose side is 5 feet long, and which therefore contains 25 square feet.

37. In Art. 33 we have proved that in the case of a rectangle

$$length \times breadth = area;$$

therefore

$$length = area \div breadth;$$

and

$$breadth = area \div length.$$

Hence when the area of a rectangle is known, and also one of its dimensions, the other can be found by division.

Again, in Art. 35 we have shown that when the rectangle is a square,

$$(side)^2 = area;$$

whence

$$side = \sqrt{(area)}.$$

Hence when the area of a square is given, the side can be found by extracting the square root of the number which denotes the area.

The student must bear in mind that before dividing the area of a rectangle by a side, both must be expressed in terms of corresponding units. Thus if the area be given in square yards and the breadth in feet, then before dividing we must either express the area in square feet, or the breadth in yards. In the first case the length will be given in feet, and in the second case in yards.

Ex. 1. *The area of a rectangular piece of cloth is 7 sq. ft. 72 sq. in., and the length is 40 in. Find the breadth.*

Here 7 sq. ft. 72 sq. in. = 1080 sq. in.

$$\text{The breadth} = \text{area} \div \text{length}$$
$$= 1080 \div 40 = 27 \text{ in.}$$
$$= 2 \text{ ft. } 3 \text{ in.}$$

Or we may proceed thus:—

$$\text{The area} = 7\tfrac{1}{2} = \frac{15}{2} \text{ sq. ft., and the length} = \frac{10}{3} \text{ ft.}$$

Hence

$$\text{the breadth} = \frac{15}{2} \div \frac{10}{3} = \frac{9}{4} \text{ ft.}$$
$$= 2 \text{ ft. } 3 \text{ in., as before.}$$

Ex. 2. *Find the side of a square whose area is 9 sq. chains, 3025 sq. links.*

We have 9 sq. chains. 3025 sq. links = 93,025 sq. links.

The square root of 93,025 is 305. Thus the length of the side is 305 links, or 3 chains, 5 links.

Ex. 3. *The area of a square is 101 sq. ft.; find the side.*

The square root of 101 cannot be found exactly; but by proceeding to three places of decimals we get 10.049. Thus the side of the square is very nearly 10.05 ft.

In cases like this we cannot get the exact answer, but by taking a sufficiently large number of decimal places we can arrive as near the true result as we please.

EXERCISES XI

Find the areas in square yards, feet, and inches of the following rectangles, whose sides are :—

1. 3 yds. 2 ft. by 2 yds. 1 ft.
2. 23 ft. by 11 ft. 4 in.
3. 11 ft. by 2 yds. 1 ft.
4. 3 yds. 2 ft. 1 in. by 5 yds.
5. 3 ft. 11 in. by 3 ft. 9 in.
6. 17 yds. 2 ft. 8 in. by 5 yds. 10 in.
7. 123 yds. 7 in. by 25 yds. 2 ft.
8. 2 ft. 5 in. by 85 ft. 9½ in.
9. 7 yds. 1 ft. 6 in. by 7½ in.
10. 62 yds. by 54 in.
11. 59 yds. 1½ in. by 40 yds.
12. 4 ft. by 2 ft. 3½ in.

Find the length of the following rectangles :—

13. Area 1080 sq. yds., and breadth 27 yds.

14 Area 15 sq. yds. 4 sq. ft., and breadth 1 yd.

15. Area 1 sq. yd. 100⅞ sq. in., and breadth 1 yd. 1¼ in.

Find the breadth of the following rectangles :—

16. Length 220 yds., and area 1 acre.

17. Length 2 yds. 2 ft. 7 in., and area 8 sq. yds. 35 sq. in.

18. Length 12 yds. 1 in., and area 4 sq. yds. 120¼ sq. in.

Find in acres, etc., the areas of the following rectangular fields, whose dimensions are :—

19. 5 ch. 3 lks. by 2 ch. 1 lk. 20. 8 ch. 9 lks by 3 ch. 11 lks.

21. 11 ch. 73 lks by 98 lks. 22. 101 ch. 3 lks. by 42 ch. 54 lks.

Find the areas in square yards, feet, and inches of the following squares, whose sides are :—

23. 1 yd. 1 ft. 1 in. 24. 17 ft. 8 in. 25. 2 yds. 1 ft. 7 in.

26. 82 ft. 9½ in. 27. 13 yds. 11 in. 28. 2 ft. 11½ in

Find the areas in acres, etc, of squares, whose sides are :—

29. 3 ch. 85 lks. 30. 1005 lks. 31. 4 ch. 8.75 lks.

32. 11 ch. 68 lks. 33. 3 ch. 16 23 lks. 34. 41 ch. 53 lks.

Find the sides of squares, whose areas are :—

35. 3481 sq. ft. 36. 10203 0201 sq. yds. 37. 10 acres.

38. 1 sq yd 2 sq ft. 138¼ sq. in. 39 10 sq. miles, 360 acres.

40 A plank is 10 in. wide; what length must be cut off that the area may be 10 sq. ft.?

41. The area of a square field is $2\frac{78}{121}$ acres; find the side of another square field of half the size.

42. Show by means of a diagram that a square yard contains 9 sq. feet

43. A rectangular field containing 1½ acres is 66 yds. broad. Show how it may be divided into 4 square plots, and find the areas of these plots.

44. Find in miles the length of gold lace, an inch wide, which would be required to cover a field of 1 acre.

38. In order to find how many times one area is contained in another, we must express them both in terms of the same denomination and then divide.

Ex. *How many bricks measuring $9\frac{3}{4}$ in. by $3\frac{3}{4}$ in. will be required to cover the floor of a room which is 52 ft. by 35 ft.?*

Here we have to find how many times the area of the face of a brick is contained in the area of the floor.

We have

$$\text{the area of a brick } = \frac{39}{4} \times \frac{15}{4} \text{ sq. in.};$$

and $$\text{the area of the floor} = 52 \times 35 \text{ sq. ft.} = 52 \times 35 \times 144 \text{ sq. in.}$$

$$\text{Hence the required number of bricks} = 52 \times 35 \times 144 \div \frac{39 \times 15}{4 \times 4}$$

$$= \frac{52 \times 35 \times 144 \times 16}{39 \times 15}$$

$$= 7168.$$

39. The area of a piece of cloth which will cover the floor of a room must be the same as the area of the floor. Therefore

$$(\textit{width of cloth}) \times (\textit{length}) = \textit{area of floor};$$

hence *to find the length of cloth required to cover the floor of a room, we divide the area of the floor by the width of the cloth.*

Ex. *How many yards of floorcloth 27 in. wide will be required for a room which is 22 ft. by 18 ft.?*

$$(\text{The required number of yards}) \times \frac{27}{36}$$

$$= \text{number of square yards in the floor}$$

$$= \frac{22 \times 18}{9}.$$

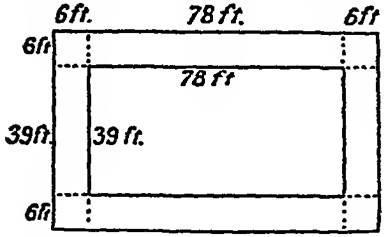
$$\text{Therefore the required number of yards} = \frac{22 \times 18}{9} \div \frac{27}{36}$$

$$= \frac{22 \times 18 \times 36}{9 \times 27}$$

$$= 58\tfrac{2}{3}.$$

40. If we make a uniform path all round the outside of a rectangular field we shall obtain another rectangle, each

side of which is greater than the corresponding side of the field by twice the breadth of the path.



Ex. *A gravel path 6 ft. wide runs round a tennis-court which is 78 ft. by 39 ft.; find the cost of making it at 5 annas per 100 sq. ft.*

[In the above diagram, for the sake of distinctness, the breadth of the path is much exaggerated.]

The area of the path is evidently the difference of the areas of the two rectangles.

Now, the length of the outer rectangle $=78+12=90$ ft.;
the breadth $=39+12=51$ ft.
Therefore the area of the outer rectangle $=90\times 51=4590$ sq. ft.;
and the area of the inner rectangle $=78\times 39=3042$ sq. ft.
Whence the area of the path $=4590-3042=1548$ sq. ft.
The cost of construction $=15.48\times 5$ annas
$=77.4$ annas
$=4$ rupees, 13 annas, 5 pies *nearly*.

41. DEFINITION.—*The whole length of the boundary of a plane figure is called its* **perimeter.** Since the opposite sides of a rectangle are equal, *its perimeter is equal to twice the sum of its length and breadth.*

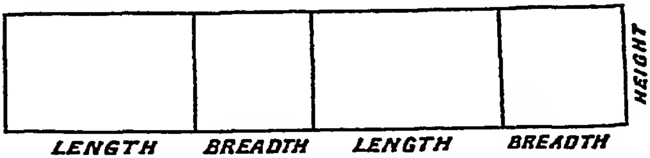
Ex. *The length and breadth of a rectangle are 11 ft. 6 in. and 10 ft. 6 in. respectively; find the area of a square of equal perimeter.*

The perimeter of the rectangle $=2$(10 ft. 6 in. + 11 ft. 6 in.)
$=44$ ft.

Hence each side of the square $=44\div 4=11$ ft.; and therefore its area $=121$ sq. ft.

42. *The area of the four walls of a room is equal to the area of a rectangle whose length is the perimeter of the room, and whose breadth is the height of the room.*

For, imagine the room to be made of cardboard, and cut



across its height through a corner, then it can be spread out into a rectangle as shown in the figure. Hence

$$\begin{aligned} \textit{area of the walls} &= \textit{height} \times 2(\textit{length} + \textit{breadth}) \\ &= \textit{height} \times \textit{perimeter}. \end{aligned}$$

Ex. *A room is* 18 *ft. long,* 16 *ft. broad, and* 12 *ft.* 6 *in. high. There is a door measuring* 7 *ft.* 6 *in. by* 4 *ft.* 6 *in., and a window* 5 *ft. by* 3 *ft.* 3 *in. Find the cost of whitewashing at* 2 *annas per* 100 *sq. ft.*

The area of the four walls $= \frac{25}{2} \times 2(18 + 16)$ sq. ft.

$= 850$ sq. ft.

Area of the door $= \frac{15}{2} \times \frac{9}{2}$ sq. ft. $= 33\frac{3}{4}$ sq. ft.

Area of the window $= 5 \times \frac{13}{4}$ sq. ft. $= 16\frac{1}{4}$ sq. ft.

Whole area to be deducted $= (33\frac{3}{4} + 16\frac{1}{4}) = 50$ sq. ft.
Area to be whitewashed $= 850 - 50 = 800$ sq. ft.
Cost of whitewashing $= 8 \times 2$ annas $= 1$ rupee.

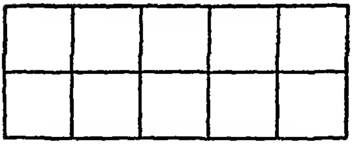
43. If the ratio of the sides of a rectangle be given, and also its area, the length of the sides can be found by dividing it into a number of squares.

Ex. *The sides of a rectangular field of* 1 *acre are in the ratio* 2 : 5; *find the length and breadth.*

Take any convenient length as unit, and construct a rectangle whose length is 5 units and breadth 2 units.

Divide the length and breadth into 5 and 2 equal parts respectively, and through the points of division draw lines parallel to the sides.

The rectangle is now divided into 10 equal squares, and therefore the area of one of these squares



$=4840 \div 10 = 484$ sq. yds.

Hence the side of a square $=22$ yds. But a side of a square is contained five times in the length and twice in the breadth of the rectangle. Therefore the length is 110 yds., and the breadth is 44 yds.

44. The student must have noticed the frequent use we have made of diagrams to illustrate the solution of examples. In working out questions in this subject, he will find that when the method to be followed is not obvious, a good drawing will often suggest it.

EXERCISES XII

1. Find how many bricks measuring 10 in. by $4\frac{1}{2}$ in. will be required for the floor of a room which is 27 ft. by 20 ft.

2. How many orange-trees can be planted on an acre of ground, when each tree requires for its growth 180 sq. ft.

3. Find the number of pupils who can be seated in an examination hall which is 52 ft. by 30 ft., if 24 sq. ft. of space is allowed for each pupil.

4. Find the expense of paving a street half a mile long and 6 yds. broad with bricks measuring $7\frac{1}{2}$ in. by 4 in., when the bricks are sold at Rs.5 per thousand.

5. How many planks 10 ft. long by $10\frac{1}{2}$ in. wide will be required for the floor of a room which is 21 ft. by 15 ft.? Find also the expense, if each plank cost 2s. 9d. and the charge for fitting be 1s. 9d. per 100 sq. ft.

6. A railway platform 120 ft. long is paved with stones 2 ft. long, 1 ft. wide, at Rs.87, 8a. per 100 stones, at a cost of Rs.1050; what is the width of the platform?

7. How many square yards of pavement are there in a street 990 yds. long and 10 ft. broad?

How many yards of carpet will be required for each of the rooms having the following dimensions?—

8. 24 ft. by 16 ft.; the carpet being 2 ft. 3 in. wide.
9. 22 ft. by 18 ft.; the carpet being 1 yd. wide.
10. 6 yds. by 5 yds.; the carpet being 30 in. wide.
11. 18 yds. by 26 ft. 3 in.; the carpet being 54 in. wide.
12. 25 ft. by 21 ft.; the carpet being 60 in. wide.

Find the cost of carpeting the following rooms:—

13. 24 ft. by 18 ft.; carpet 54 in. wide at Rs.10 per yd.
14. 26 ft. by 17 ft. 6 in.; carpet 30 in. wide at Rs.2, 8a. per yd.
15. 17 ft. 9 in. by 12 ft. 2 in.; carpet 2 ft. wide at 4s. 6d. per yd.
16. 18 ft. by 18 ft.; carpet 29 in. wide at Rs.5 per yd.
17. 20 ft. 9 in. by 24 ft. 3 in.; carpet 2 ft. wide at Rs.2, 10a. per yd.
18. 5 yds. by $4\frac{1}{2}$ yds.; at 10s. per sq. yd.
19. 32 ft. by 22 ft. 6 in.; at Rs.12, 8a. per sq. yd.
20. 45 ft. by 37 ft. 6 in.; at Rs.11, 4a. 6p. per sq. yd.
21. The cost of carpeting a room 21 ft. long at Rs.2, 8a. per sq. yd. is Rs.105; find the breadth of the room.
22. Find the width of the carpet, when the cost of carpeting a room which is 24 ft. by 18 ft., with carpet at Rs.2, 12a. per yd., is Rs.176.
23. What is the highest price that can be paid per yd. for matting, which is 27 in. wide, in order that the expense of matting a room measuring 45 ft. by 36 ft. may not exceed Rs.5?
24. The sides of two squares are 28 ft. and 96 ft.; find the side of a square which is equal to the sum of the two.
25. The sides of two squares are 65 ft. and 60 ft.; find the side of a square which is equal to the difference of the two.
26. The length of a rectangular field is 117 links, and the breadth is 83 links; find the area of a square of equal perimeter.
27. Each side of an equilateral triangle is 4 ft.; find the area of a square of equal perimeter.
28. The sides of a rectangle are in the ratio of 5 : 7, and the area is 2240 sq. yds.; find the length and breadth.
29. The sides of a triangle are in the ratio of 3 : 4 : 5, and the perimeter is 143 ft.; find the lengths of the sides.
30. The sides of two squares are in the ratio of 2 : 3; compare their areas.
31. The sides of a rectangle are in the ratio of $\frac{2}{3} : \frac{3}{4}$, and the area is 1800 sq. ft.; find the lengths of the sides.
32. A rectangular courtyard 96 ft. long and 60 ft. broad requires for its pavement 20,736 bricks similar in shape to the courtyard; find the dimensions of a brick.

Find the length of wall-paper required for the following rooms :—

33. 22 ft. long, 18 ft. broad, 17 ft. 6 in. high ; paper 21 in. wide.

34. 18 ft. long, 16 ft. broad, 17 ft. 6 in. high ; paper 21 in. wide.

35. 16 ft. long, 9 ft. broad, 12 ft. 9 in. high ; paper 1 yd. wide.

36. 25 ft. 6 in. long, 18 ft. broad, 20 ft. 3 in. high ; paper 27 in. wide.

37. 18 ft. 9 in. long, 12 ft. 3 in. broad, 15 ft. high ; paper 25 in. wide.

38 Find the cost of papering a room 23 ft. long, 17 ft. 8 in. wide, and 15 ft. 6 in. high, with paper 16 in. wide, at 10s. 6d. per piece of 12 yds.

39. Find the expense of papering a room 32 ft. 6 in. long, 20 ft. 6 in. wide, and 18 ft. high, with paper 27 in. wide, at 5s. 4d. per piece of 12 yds.

40. Find the cost of papering a room 20 ft. long, 16 ft. wide, and 13 ft. 6 in. high, with paper 21 in. wide, at 3a. 6p. per yard ; an area of 100 sq. ft. being deducted for doors.

41. Find the expense of papering a room whose length is 35 ft., breadth 21 ft., and height 18 ft. 6 in., with paper 21 in. wide, at 7s. 6d. per piece of 12 yds. ; deducting for two doors 7 ft. 6 in. by 4 ft. 6 in., three windows 5 ft. by 3 ft. 6 in., and a fireplace 6 ft. by 5 ft.

42. A room is 33 ft. 9 in. long, 22 ft. 3 in. broad, 20 ft. high : find—

(i.) Cost of floorcloth, 3 ft. 6 in. wide, at 3a. a yard.

(ii.) Cost of whitewashing the walls at 2a. per 100 sq. ft.

(iii.) Cost of painting the ceiling at 5a per sq. yd.

43. A tennis-court is 78 ft. long, 39 ft. broad, and has a gravel path 4 ft. 6 in. wide outside it. Find the area of the path.

44. A square park is 1000 yds. long. A road 12 ft. broad is made all round it inside ; find its area.

45. An oblong garden is 800 yds. long and 450 yds. broad. Two roads, each 10 ft. wide, pass through the centre of the garden, dividing it into four equal rectangular plots. Find the cost of constructing the roads at R. 1, 2a. per 100 sq. ft.

46. A room 21 ft. long, 18 ft. broad, has a floorcloth 2 ft. 6 in. wide all round as a border ; the rest of the floor is covered by a carpet costing Rs. 9, 12a. per sq. yd. ; the floorcloth costs 3a. 6p. per yd. Find the total cost.

47. A square field of one acre has a square tank in the centre whose side is 22 yds. ; find the area under cultivation.

48. The rent of a square field at Rs 6, 8a. per acre is Rs. 260 ; find the cost of planting a cactus hedge round it at 5a. 3p. per chain.

49. Find the cost of lining the inside of a box 4 ft. 6 in. long, 4 ft. wide, and 3 ft. 6 in. deep with tin, weighing $1\frac{1}{2}$ lbs. to the square yard, at $10\frac{1}{2}$d. per lb.

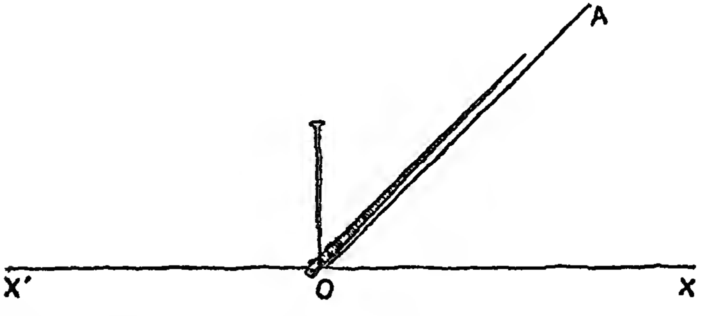
50. The length of a hall is three times the breadth; the cost of whitewashing the ceiling at $5\frac{1}{3}$d. per sq. yd. is £4, 12s. $7\frac{1}{2}$d., and the cost of papering the four walls at 1s. 9d. per sq. yd. is £35. Find the height of the hall.

BOOK II

CHAPTER V

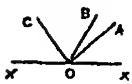
ANGLES

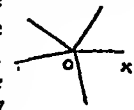
45. Rule a line XOX', and place a large needle to coincide with OX, having its eye at the point O. Fix the eye at O with another fine needle, and turn the large needle into the position OA. Rule the line OA, and give the



needle another turn until it coincides with OX'. The needle has described the two angles AOX and AOX', but from OX to OX' it has been turned through two right angles (Art. 11). Hence we conclude that *the angles which one straight line makes with another straight line, on one side of it, are together equal to two right angles.*

In the same way we can demonstrate that *the sum of all the consecutive angles which can be formed round the point* O *in the line* XX′, *and on one side of* XX′, *is equal to two right angles.* By making a needle start from *OX* and travel round the point *O* back again into the position *OX*, we can show that *the sum of all the consecutive angles which can be formed round a point is equal to four right angles.*



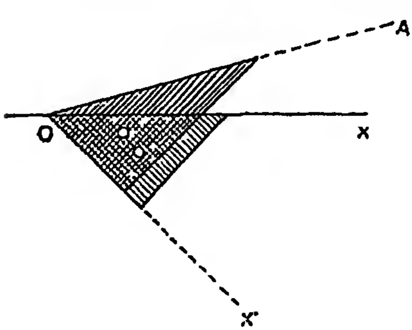


DEFINITION.—*When the sum of two angles is equal to two right angles, each is said to be the* **supplement** *of the other, and the two angles are said to be* **supplementary.**

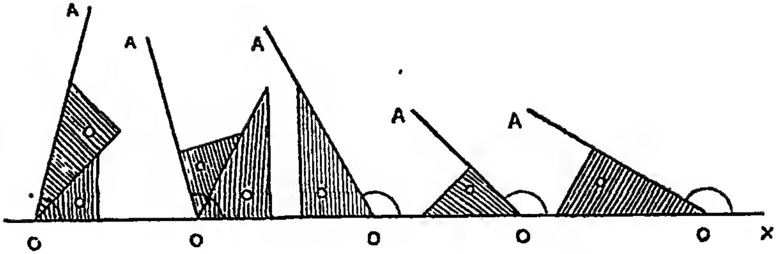
Thus in the first figure the angles *AOX* and *AOX′* are supplementary, so that if one arm of an angle be produced, the other arm makes with the produced line the supplement of the angle.

Since two right angles contain 180°, the supplement of 30° is an angle of 180° − 30°, or 150°. Similarly the supplements of angles of 45° and 60° are angles of 135° and 120° respectively.

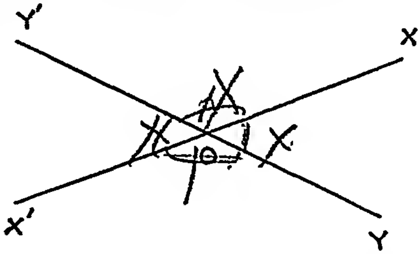
46. Rule a line *OX*, and place your 45° set-square as shown in the figure. Rule the line *OX′*, remove the first set-square, and place the 60° set-square with the shortest edge in coincidence with *OX′*. Now rule the line *OA*, then the angle *AOX* thus constructed is of 15°.



The student can now construct the angle *AOX* of 75°, 105°, 120°, 135°, and 150° by placing the set-squares as shown in the following figures:—

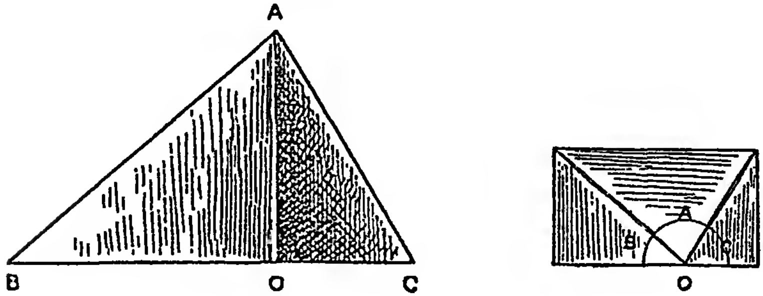


47. Draw two straight lines XX', YY' cutting one another in O.



Cut out the angles XOY, $X'OY'$. Fit them together to find out whether they are equal. Try the same thing with several pairs of lines and you will find that in every case the angles XOY, $X'OY'$ are equal, as also the angles XOY', $X'OY$. We therefore conclude that *if two straight lines cut one another, the vertically opposite angles are equal.*

48. Draw any triangle ABC, and let BC be the side



which is not less than either of the other two. Draw AO perpendicular to BC, which can be done by folding.

Cut out the triangle ABC and fold it into the shape shown in the second figure. This will be done by bringing the points A, B, and C successively to O and making creases. Now the three angles of the triangle ABC appear in the second figure as forming two right angles (Art. 45). By repeating this experiment with a number of triangles we conclude that—

The sum of the three angles of any triangle is equal to two right angles.

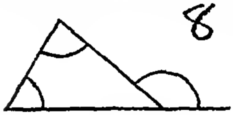
EXERCISES XIII

1. How many degrees are there in the three angles of any triangle?

2. Draw a triangle, and with your protractor measure all its angles in degrees, and add them together. The sum ought to be equivalent to two right angles. (Repeat this for several triangles.)

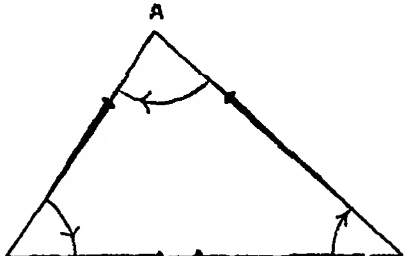
3. Prove that the sum of any two angles of a triangle is the supplement of the third.

4. Draw any triangle, and cut off all its angles. Place them in such a way as to show that they together make two right angles.



5. *If a side of a triangle is produced the exterior angle is equal to the sum of the two interior opposite angles.* Prove this by applying Ex. 3, or after the manner of Ex. 4.

6. Make a triangle ABC. Place a pin along the side BC, and make it travel along all the sides and back again to BC by turning it through the angles C, A, and B, as indicated by the arrow heads in the figure. Now, looking at the starting and finishing positions of the pin in BC, we notice that it is turned through two right angles. But it has also been turned through all the angles of the triangle. What can we infer from this?



7. Two angles of a triangle are $57° 36' 47''$ and $62° 31' 22''$ respectively; what is the third angle equal to?

8. Show that a triangle cannot have more than one right angle or one obtuse angle, but that it may have three acute angles.

9. If one angle of a triangle be equal to the sum of the other two, the triangle will be right-angled. Illustrate with your set-squares.

10. One angle of a triangle contains $36°$, and the other two are equal; show that each of these angles is double of the first.

11. Two angles of a triangle are $32° 46' 49''$ and $57° 13' 11''$; show that the triangle is right-angled.

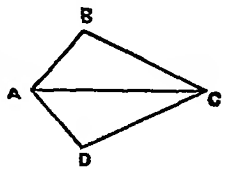
12. One of the acute angles of a right-angled triangle contains $30°$; show that the other acute angle is two-thirds of a right angle.

13. Take AB four inches long; draw BC at right angles to AB and three inches in length; join AC and produce it to X. Verify by measurement that the angle BCX is equal to the sum of the angles at B and A.

14. Repeat the last exercise by taking AB, BC equal to 10 and 7 cm. respectively.

15. Draw a triangle ABC, and let BC be its longest side; draw AL perpendicular to BC. You have now two right-angled triangles; measure their acute angles, and verify that in each case the sum of the acute angles is equal to a right angle. What do you infer about the sum of the angles of the triangle ABC?

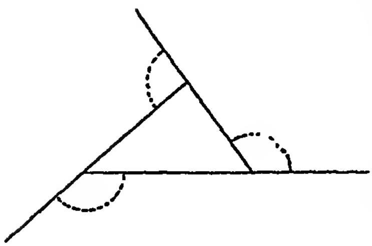
49. Draw any quadrilateral $ABCD$, and rule the diagonal AC. It is evident from the figure that the sum of the angles of the quadrilateral is the same as the sum of the angles of the two triangles into which it is divided by the diagonal. Hence—



The sum of the angles of a quadrilateral is equal to four right angles.

50. Draw a triangle and produce the sides, going round the triangle in one direction; then at each angular point of the triangle we have an interior angle of the triangle, and also an exterior angle, marked on the figure by a

dotted arc, and the sum of these two is equal to two right angles (Art. 45). Hence the sum of all the interior and exterior angles is equal to six right angles; but the interior angles together are equal to two right angles; hence



All the exterior angles of a triangle formed by producing the sides taken in order are together equal to four right angles.

EXERCISES XIV

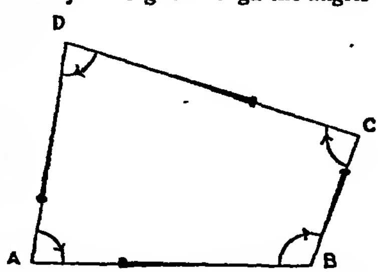
1. Draw three quadrilaterals of different shapes, and measure their angles in degrees. What is the sum of the angles in each case?

2. Draw AC 3″ long, and at A and C construct angles of 60° and 30° respectively, both above and below the line AC. You thus obtain the quadrilateral $ABCD$. Measure the angles at B and D, also the the sides AB and AD.

Find O the middle point of AC, and discover what kind of a quadrilateral is $ABOD$?

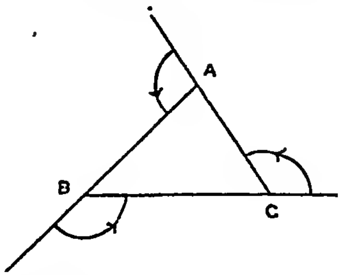
3. Cut off all the angles of a quadrilateral, and fit them together round a point. Is the angular space about the point exactly filled up? What do you infer from this fact?

4. Draw any quadrilateral $ABCD$. Place a pin along the side AB, and make it travel round all the sides by turning it through the angles of the quadrilateral as shown in the figure. It has gone once right round and arrived at the same place and position as it had at starting. In doing so, it has been turned through the angles of the quadrilateral in the same clock-wise direction throughout. What conclusion can you draw from this fact?



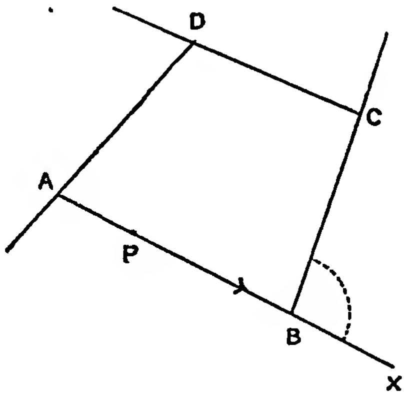
5. Draw a triangle, and produce its sides going round the triangle

in one direction. Place a pin along the side BC, and make it travel along all the sides, turning it through the exterior angles as shown in the figure. What is its position at the finish? What do you infer from it?



6. A man starts from the point P, and travels round the sides $ABCD$ of a quadrilateral field, arriving again at the point P. Through what angle has he turned?

Notice that when the man arrives at B his direction of motion is along BX, therefore when he commences to walk along BC he turns to his left through the angle XBC, *i.e.* through an exterior angle of the quadrilateral.



7. If a man walks round a circle back to his starting point, what angle does he turn through?

8. From the last two exercises what do you infer about the exterior angles of a quadrilateral formed by producing the sides taken in order? And what about the exterior angles of any figure contained by straight lines?

9. Apply the method of Art. 50 to prove that the exterior angles of any quadrilateral are together equal to four right angles.

10. Draw a triangle and a quadrilateral. Produce their sides taken in order, measure all the exterior angles in each case and find their sum.

11. Draw any quadrilateral $ABCD$; take a point O within it and

join it to the angular points. The quadrilateral is now divided into four triangles. What is the sum of the angles of these four triangles? What is the sum of the angles at *O*? (Art. 45). Hence what do you infer about the sum of the angles of the quadrilateral?

12. After the manner of the last example show that the sum of the angles of a five-sided figure is

$$2 \times 5 - 4, \text{ or } 6 \text{ right angles.}$$

13. Show that the sum of the angles of a six-sided figure is

$$2 \times 6 - 4, \text{ or } 8 \text{ right angles.}$$

14 If all the angles of a six-sided figure are equal, show that each angle is of 120°.

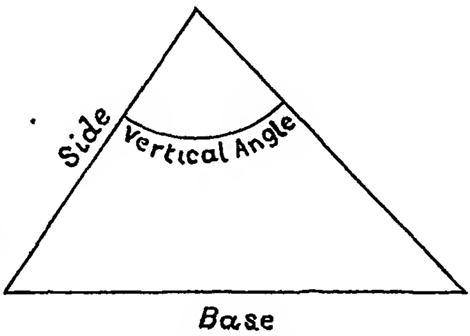
15. Two of the angles of a quadrilateral are right angles, show that the other two angles are supplementary.

CHAPTER VI

TRIANGLES

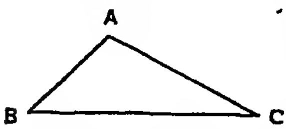
51. THE simplest of all the figures contained by straight lines is a triangle. It has three sides and three angles, and we shall see that when any three of these six elements, except the three angles, are given, the triangle can be constructed.

Any side of a triangle, usually the lowest in a drawing, may be called its **base**, then the other two are its **sides**.



The angle opposite to the base is called the **vertical angle**, and the angles adjacent to the base are called **the angles at the base**.

52. Draw five triangles of different shapes, and measure their sides. You will find that in every case the sum of the lengths of any two sides is greater than the length of the third side. Hence

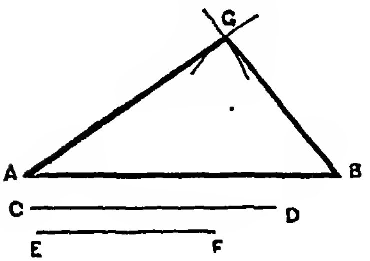


Any two sides of a triangle are together greater than the third.

This is otherwise evident, for the straight path from B to C must be shorter than the crooked path BA, AC.

53. *To construct a triangle, its three sides, AB, CD, and EF being given.*

With B as centre, and radius CD, describe an arc; with A as centre, and radius EF, describe another arc to intersect it in G. Join AG and BG.



The arcs will not intersect unless CD and EF are together greater than AB (Art. 52).

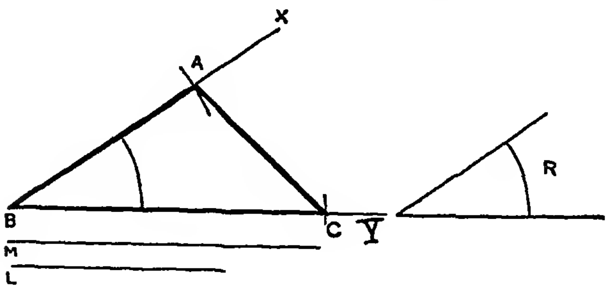
Let the student make the above construction for the lengths 4″, 2″, and 1″, taking AB in succession equal to each of these lengths, and CD and EF equal to the remaining lengths. He will find that the construction fails in every case.

Ex. Construct triangles whose sides are given below, making two of each kind; cut them out and fit together each pair, and see if they are exactly equal to one another—

(i.) 3″, 4″, 5″. (ii.) 2.7″, 3″, 1″. (iii.) 2.8″, 2.5″, 2.3′. (iv.) 6 cm., 7 cm., 8 cm. (v.) 2″, 3″, 2″. (vi.) 5 cm., 5 cm., 5 cm.

54. *To construct a triangle, two sides and the angle between them being given.*

Make the angle XBY equal to the given angle R, and



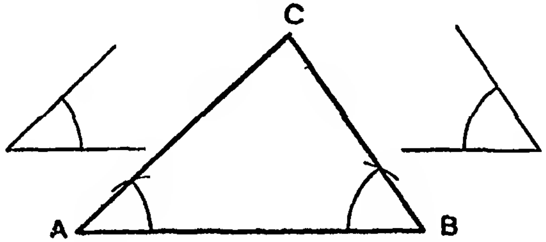
cut off BA, BC equal to the given sides L and M. Join AC. Then ABC is the required triangle.

Ex. Construct the following triangles, two sides and the included angle being given in each case; make two of each kind, cut them out, and fit together each pair, and find out if they are exactly equal to one another:—

(i.) 2″, 4″, 60°. (ii.) 2.8″, 2.3″, 54°. (iii.) 5 cm., 7 cm., 132°. (iv.) 4″, 4″, 36°. (v.) 6 cm., 6 cm., 60°. (vi.) 1.9″, 2.3″, 150°.

55. *To construct a triangle, being given the base and the angles at the base.*

Let AB be the given base. At A and B make angles



equal to the given angles on the same side of AB. Let the containing lines meet in C. Then ABC is the triangle required.

Ex. 1. Make two of each kind of the following triangles, the base and the angles at the base being given; cut them out and fit together each pair, and discover if they are equal to each other in every respect:—

(i.) 3″, 36°, 54°. (ii.) 2.7″, 40°, 66°. (iii.) 6 cm., 105°, 30°. (iv.) 4.2″, 45°, 72°. (v.) 7 cm., 36°, 36°. (vi.) 3.2″, 60°, 60°.

Ex. 2. In a triangle ABC the base AB is 3″, the angle at A is of 40°, and the angle at C of 66°. Construct the triangle in duplicate, and see if the two are exactly equal.

56. From the exercises of the last three articles we see *that two triangles are equal in all respects when the following parts are severally equal:—*

(i.) *The three sides.*

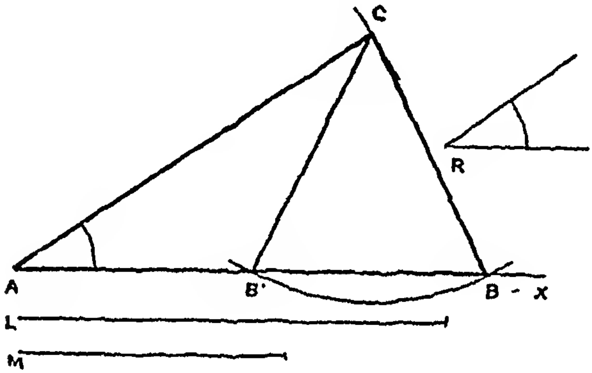
(ii.) *Two sides and the angle contained by them.*

(iii.) *Two angles and the adjacent side.*

(iv.) *Two angles and the side opposite one of them.*

Triangles which are equal in every respect are said to be **congruent**.

57. *To construct a triangle, being given two sides and the angle opposite one of them.*



Let L and M be the given sides, and R the angle opposite to M.

Rule a line AX, and at A make the angle CAX equal to R. Cut off AC equal to L. With centre C, and a radius equal to M, describe a circle. This circle will in general cut the line AX in two points B, B'. Join CB, CB'. Then either of the triangles CAB, CAB' satisfies the required conditions.

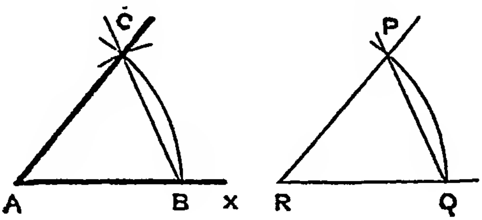
The circle may meet the line AX in one point only, in that case only one triangle can be made; and it may happen that the circle does not meet the line at all, in that case no triangle can be constructed with the given parts.

Ex. Construct the triangle ABC, having given—

(i.) $CA=4''$, $BC=3''$, $\angle A=36°$. (ii.) $CA=7$ cm., $CB=3.5$ cm., $\angle A=30°$. (iii.) $CA=3.5''$, $BC=1.3''$, $\angle A=45°$. (iv.) $CA=3.2''$, $BC=5.1''$, $\angle A=40°$. (v.) $CA=4''$, $BC=2''$, $\angle A=30°$. (vi.) $CA=5$ cm., $BC'=2.3$ cm., $\angle A=75°$. In (ii.) and (v.) measure the angle at B.

58. *At the point A in the straight line AX to make an angle equal to the given angle R.*

Cut off RP equal to RQ, and join PQ. With centre



A, and radius equal to RP, describe a circle cutting AX in B. With centre B, and radius QP, describe another circle cutting the first in C. Join AC, BC. Then CAX is the required angle.

For in the two triangles PQR, CAB, the three sides of the one are equal respectively to the three sides of the other. Hence the triangles are congruent, and the angle at A is equal to the angle R (Art. 56).

Show that if the circles whose centres are A and B are completed you at once get two angles each equal to R; one on each side of AB.

Hence make an angle double of a given angle.

EXERCISES

Construct the triangle ABC, having given :—

1. $AB=13$ cm., $BC=14$ cm., $CA=15$ cm.
2. $AB=4.5''$, $BC=4.2''$, $CA=4.8''$.
3. $AB=BC=CA=7.5$ cm. Measuie all the angles.

4. $AB = BC = CA = 3.2''$. Measure the angles.

5. $AB = CA = 4''$, $BC = 2.8''$. Measure the angles at B and C. Are they equal?

6. $AB = CA = 3.5''$, $BC = 2.5''$. Measure the angles at B and C. Are they equal?

7. $AB = CA = 6$ cm., $BC = 3.5$ cm. Measure the two largest angles. Are they equal?

8. $AB = 3''$, $CA = 2.5''$, $BC = 4''$. Measure the angles. Is the largest angle opposite to the longest side? Is the smallest angle opposite to the shortest side?

9. $AB = 3''$, $BC = 4''$, $\angle B = 60°$.

10. $AB = 5$ cm., $CA = 7.5$ cm., $\angle A = 75°$.

11. $BC = 3.8''$, $CA = 4.2''$, $\angle C = 105°$.

12. $AB = BC = 3.7''$, $\angle B = 52°$. Measure the other angles.

13. $AB = CA = 8$ cm., $\angle A = 36°$. Measure the other angles.

14. $BC = CA = 4.3''$, $\angle C = 60°$. Measure the remaining angles and the third side. All the angles of this triangle are equal, and all the sides are also equal.

15. $BC = 3.6''$, $\angle B = 30°$, $\angle C = 60°$. Measure the third angle and the other sides, and show that the smallest side is exactly half of BC.

16. $CA = 7$ cm., $\angle C = \angle A = 75°$. Measure the third angle, and show that the other two sides are equal.

17. $AB = 4.2''$, $\angle A = \angle B = 60°$. Measure the remaining angle and sides of the triangle. Are all the sides equal?

18. $BC = 3.2''$, $\angle B = 60°$, $\angle A = 45°$.
(*First find the angle C by Art.* 48.)

19. $CA = 5.8$ cm., $\angle B = 36°$, $\angle A = 54°$.

20. $AB = 4''$, $\angle C = \angle A = 60°$.

21. $AB = 5''$, $AC = 2.5''$, $\angle B = 30°$.

22. $AB = 4''$, $AC = 3''$, $\angle B = 36°$.

23. Show by means of a figure that it is impossible to construct a triangle ABC in which $AB = 4''$, $AC = 1.7''$, and $\angle B = 30°$.

24. Draw an angle of 54° using the protractor. Make a copy of it by the construction of Art. 58.

25 Draw an angle of 36°, and by construction make an angle equal to it.

26 Draw an angle of 32°, and by construction make an angle twice as great. Check your construction by measuring with the protractor.

27. Construct a triangle whose sides are $3.5''$, $2.8''$, $4.2''$. Construct another triangle on a base $3''$ long whose angles are equal to the angles of this triangle.

28. Construct a triangle whose base is 4″, and of the remaining sides one is twice as long as the other, the sum of these sides being 8·1″.

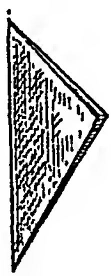
29. The base of a triangle is 3″, and each of the angles at the base is double of the third angle; construct the triangle.

30. Construct a triangle on a base 4″, such that the angle at the vertex is twice as large as either of the angles at the base.

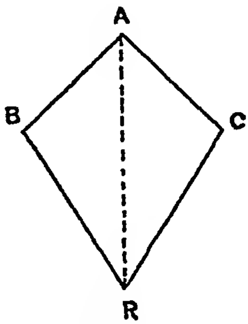
Is it like one of your set-squares?

59. *To bisect a given angle A.*

Draw an angle on paper, and cut it out. Fold the paper so that the lines containing the angle lie on each other. Cut off the angle so that the crease forms the longest side of the double triangle cut off. Now unfold the paper. From (1) it is evident that the crease is the bisector of the angle, and from (2) we see that we must



(1)

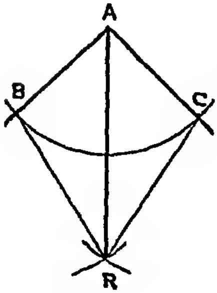


(2)

make the arms *AB* and *AC* equal, and then find *R* such that *CR* and *BR* are equal. Hence we must proceed as follows:—

With *A* as centre, and any radius, describe an arc cutting off *AB* equal to *AC*. With *B* and *C* as centres, and any the same radius, describe arcs intersecting in *R*. Join *AR*. Then *AR* is the bisector of the angle *A*.

In the triangles BAR, CAR the three sides of the one are respectively equal to the three sides of the other; hence the triangles are congruent, and the angle BAR is equal to the angle CAR.



EXERCISES

1. Construct angles of $22\frac{1}{2}°$, $37\frac{1}{2}°$, and $67\frac{1}{2}°$ by help of the set-squares and the above construction.

2. Divide a given angle into 4, 8, 16 . . . parts.

3. Draw an angle of 53°. Bisect it and measure its parts with the protractor.

4. Make an angle of 130°. Divide it into 4 equal parts. Point out in your figure the angle which contains $97\frac{1}{2}°$.

5. Make the figure of Art. 45, taking $AOX=45°$. Bisect the angles AOX, AOX' and measure the angle between the bisectors.

6. Repeat the last exercise, taking AOX equal to 30°, 60°, 75°, and 42° respectively.

You will find in each case that the angle between the bisectors is a right angle. Hence we conclude that—

Straight lines which bisect adjacent supplementary angles are at right angles.

7. Draw a triangle with sides 4″, 4.2″, and 4.8″. Bisect the three angles, and notice that the bisectors pass through the same point.

8. Repeat the last exercise with a triangle whose sides are 7 cm., 9.5 cm., 11.2 cm

9. Draw a parallelogram whose sides are 3.5″ and 5″. Bisect the four angles, and examine the figure enclosed by the bisectors. Is it a parallelogram?

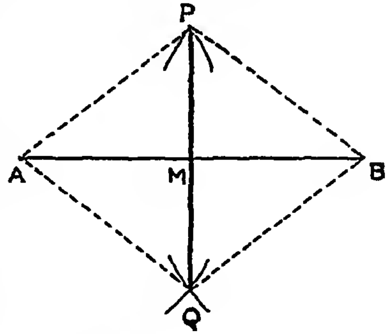
10 Draw a triangle with sides 13 cm., 14 cm., 15 cm.; draw also the bisectors of the angles of the triangle meeting in the same point O. With O as centre and a radius of 5 cm. describe a circle, and verify by measurement that this circle cuts off from each side of the triangle a portion 6 cm. in length.

60. *To bisect a given straight line AB.*

With A as centre, and any convenient radius, describe arcs

above and below the line. With B as centre, and the same radius, draw arcs, intersecting the former in P and Q.

Draw the line PQ; it will cut AB in M, its middle point. For the triangles APQ, BPQ are congruent by Art. 56; hence the angles APM and BPM are equal.



Again the triangles APM and BPM are congruent by Art. 56; therefore AM is equal to BM.

What kind of a quadrilateral is the figure $APBQ$? Make the above construction, taking AB six inches in length, and the radius AP five inches. Cut out the figure formed, and show by folding that its diagonals bisect each other at right angles.

EXERCISES

1. Take a line AB 6″ long. With centres A and B and radii 5″ describe arcs intersecting in P and Q. Join PQ, cutting AB in M.

Verify by measurement that AB is bisected in M.

2. Bisect a line 5.6″ long using radii of 4.6″.

3. Draw a line AC 3·8″ long and bisect it at right angles at M by the line BMD. Cut off $BM=DM=1.9$″. The figure $ABCD$ is a square.

4. Draw a circle of 2″ radius and rule the diameter AC. Draw the diameter BD which bisects AC at right angles, and prove by measurement that the figure $ABCD$ is a square.

5. Bearing in mind that the diagonals of a rhombus intersect at right angles, construct a rhombus whose diagonals are 6 cm. and 8 cm. in length respectively.

Measure the sides of the rhombus.

6 Construct a square whose diagonal is 4″.

7. Take AB 3.5″ in length. With centres A and B and radii 2.5″ describe arcs intersecting in P above AB, and with the same centres and radii 5″ describe arcs intersecting in Q below AB. Join PQ and

show that it bisects AB at right angles. The figure $APBQ$ is called a *kite*.

Notice that the longer diagonal of a kite bisects the shorter diagonal at right angles.

8. In the last exercise let the second pair of arcs also intersect above AB in Q. Join QP and produce it to meet AB. Show that it will bisect AB.

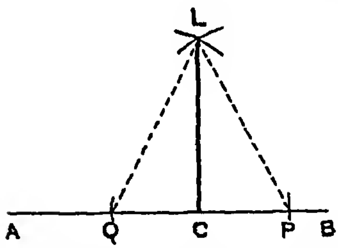
9. Take two points A, B 8 cm. apart, and draw the line which bisects AB at right angles. Show by measurement that every point on the right bisector of AB is equidistant from A and B.

10. Draw a triangle ABC with sides 4.5″, 3.8″, and 3.2″ long. Draw the right bisectors of the three sides, and notice that they all pass through the same point O. Is the point O equidistant from A, B, and C? With centre O and radius OA describe a circle and see that it passes through B and C.

11. Construct a triangle with sides $3\frac{1}{2}''$, $4\frac{1}{4}''$, $5''$ and draw the circle which passes through its angular points.

12. Draw a triangle having each side 6 cm., and describe the circle which passes through its angular points.

61. *To draw a straight line at right angles to a given straight line AB from a point C in it.*



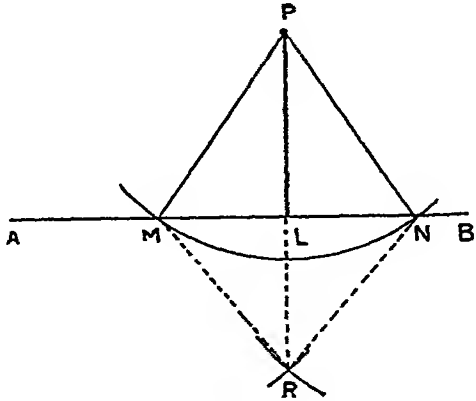
Cut off CP equal to CQ. Describe arcs having P and Q as centres and any equal radii. Let the arcs intersect in L. Join CL. Then CL is at right angles to AB. The triangles BCL, QCL are congruent by Art. 56; therefore the angles at C are right angles.

In making the above construction, it may be found necessary to produce the line AB.

Ex. Take a piece of paper with a straight edge. Fold it so that the edge is doubled on itself. Cut off the angle and unfold. Why is the crease perpendicular to the edge? Deduce the above construction from this.

62. *To draw a perpendicular to the straight line AB from a point P outside it.*

With P as centre, and a long enough radius, describe an



arc to intersect AB in M and N. With M and N as centres, and any radius greater than the half of MN, describe arcs cutting in R. Join PR, meeting AB in L; then PL will be the required perpendicular. For the triangles PMR, PNR are congruent by Art. 56, therefore the angle at P is bisected.

Again, the triangles PLM and PLN are congruent by Art. 56, therefore the angles at L are equal.

Here, also, it may be found necessary to produce the line AB, and it may also happen that the foot of the perpendicular PL falls on the produced part of the line.

EXERCISES

1. Draw a straight line AB 6 cm. long, and through its middle point draw a straight line at right angles to it.

2. Take AB 7″ long, and through a point in it 2.5″ from B draw a straight line at right angles to it.

3. Draw a line 4.5″ in length, and at one extremity of it erect a perpendicular to it.

4. Take AO 3″ long, through O draw BOC at right angles to AO

and such that $OB=OC=OA$. Show by measurement that AB is at right angles to AC.

5. Draw a line 5.6″ long, and through a point which divides it in the ratio of 4 : 3 draw a straight line at right angles to it.

6. Construct a triangle with sides 3.7″, 4″, 4.8″ and from the angular points draw perpendiculars to the opposite sides.

These perpendiculars ought to pass through the same point.

7 Draw a triangle whose base is 8 cm. and the angles at the base are of 30° and 60° respectively. Draw a perpendicular from the vertex on the base and show that this perpendicular makes angles of 30° and 60° with the two sides.

8. Construct a parallelogram $ABCD$ such that $AB=5''$, $AD=3.5''$, and the angle at $A=60°$. Join BD and draw AL, CM perpendiculars to BD. Verify that these perpendiculars are equal and that AM is parallel to CL.

9. Draw AB, AC containing an angle of 30°; cut off $AB=7.8$ cm. and draw BL perpendicular to AC. Show by measurement that the length of BL is 39 mm.

The Right-angled Triangle

63. We shall now consider three special kinds of triangles, viz. the right-angled triangle, the isosceles triangle, and the equilateral triangle.

A right-angled triangle is one which has a right angle.

In a right-angled triangle any one of the two sides containing the right angle may be called **the base**, and then the other side is called **the perpendicular**.

The side opposite to the right angle is called **the hypotenuse**.

Thus in the triangle ACB, where C is a right angle, BC is called the base, AC the perpendicular, and AB the hypotenuse.

DEFINITION.—*If the sum of two angles be equal to one right angle, the angles are called* **complementary angles**, *and each is said to be* **the complement** *of the other.*

It follows from this definition and Art. 48 that *the two acute angles of a right-angled triangle are complementary.*

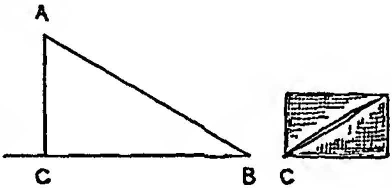
64. *To construct a right-angled triangle, having given the base and the perpendicular.*

This construction is the same as that of Art. 54, the contained angle being a right angle.

EXERCISES XV

1. Construct a right-angled triangle whose base is 3″ and perpendicular 4″. Measure the acute angles, and verify that they are complementary.

2. Construct a right-angled triangle, the base and perpendicular being 12 cm. and 5 cm. respectively. Measure the hypotenuse, and verify that the sum of the squares described on the sides containing the right angle is equal to the square described on the hypotenuse.



3. Make the triangle ABC; the angle at C being a right angle, and the sides BC, CA being 8″ and 6″ respectively. Fold the triangle double by bringing A and B to C. Does this folding show the acute angles to be complementary? Does it also prove that the middle point of the hypotenuse is equally distant from the three angular points?

4. Construct a right-angled triangle, the base and perpendicular being 3.6″ and 4.8″ respectively. Measure the hypotenuse, and also the distance between its middle point and the right angle.

Describe a semicircle on the hypotenuse as diameter. Does it pass through the right angle? Why?

5. Construct the following right-angled triangles, the bases and perpendiculars being given; in each case measure the hypotenuse, and verify that the sum of the squares on the sides containing the right angle is equal to the square on the hypotenuse:—

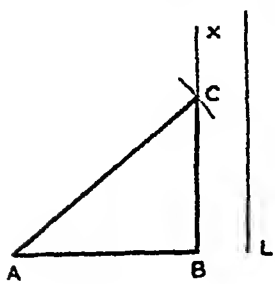
(i.) 4.5″, 2.8″. (ii.) 10.5 cm., 10 cm. (iii.) 6.5″, 7.2″. (iv.) 5.5″, 4.8″.

6. On the hypotenuses of the triangles of the last example describe semicircles, and see that they pass through the right angle.

7. Cut out the triangles of Ex. 6, and fold them after the fashion of

Ex. 3. By observing the folded triangles try and discover a rule for finding the area of a right-angled triangle, and hence find the areas of these triangles.

65. *To construct a right-angled triangle, having given the hypotenuse* (L) *and one of the sides containing the right angle* (AB).



At B draw BX at right angles to AB.

With centre A, and radius equal to L, describe an arc cutting BX in C. Join AC. Then ABC is the required triangle.

Ex. 1. Construct the following right-angled triangles, the hypotenuse and one of the sides containing the right angle being given in each case; measure the other side, and verify that the difference of the squares on the hypotenuse and a side is equal to the square on the other side:—

(i.) 4″, 3.2″. (ii.) 3.9″, 1.5″. (iii.) 2.5″, 2.4″. (iv.) 4.1″, 4″.

Ex. 2. The following sets contain three numbers, such that the sum of the squares of two of them is equal to the square of the third; construct triangles the length of whose sides, measured in inches, is equal to these numbers, and measure their greatest angles. What do you find out? Enunciate your conclusion in general terms.

(i.) 2, 2.1, 2.9. (ii.) 2.5, 6, 6.5. (iii.) 5.8, 4, 4.2. (iv.) 3.7, 1.2, 3.5.

Ex. 3. Construct the following right-angled triangles, the hypotenuse and one side being given in each case; make two of each kind, cut them out, and fit together each pair, and find out if they are exactly equal to one another:—

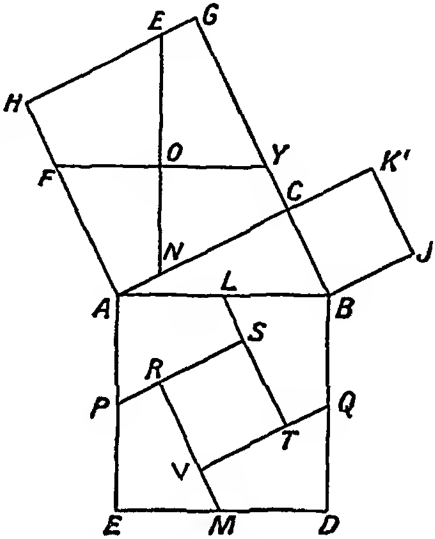
(i.) 5″, 3″. (ii.) 7 cm., 4 cm. (iii.) 13 cm., 12 cm. (iv.) 4.8″, 3.9″. (v.) 3.7″, 2.8″. (vi.) 4.7″, 3.5″.

From these exercises we see that *if two right-angled triangles have their hypotenuses equal, and one side of the one equal to one side of the other, the triangles are congruent.*

66. *To prove by dissection that the square on the hypo-*

tenuse of a right-angled triangle is equal to the sum of the squares on the other sides.

Through the point *O*, where the diagonals *CH* and *AG* of the square on the larger side intersect, draw two lines, one perpendicular and the other parallel to the hypotenuse.



Find *L*, *M*, *P*, *Q*, the middle points of the sides of the square on the hypotenuse, and through them draw *LT*, *MR* parallel to *BC* and *PS*, *QV* parallel to *AC*. These lines will form by their intersections a square *RSTV*.

On cutting up the figure, the student will find that the quadrilaterals *CNOY*, *ANOF*, *HEOF*, *GEOY*, *ALSP*, *BQTL*, *DMVQ*, *EPRM* are all equal to one another; and that the square *CKJB* is equal to the square *RSTV*.

Ex. Construct right-angled triangles whose sides containing the right angle are given below; describe squares on all the sides, and dissect them to prove the proposition of this article :—

(i.) 3″, 7″. (ii.) 5 cm., 7 cm. (iii.) 3.6″, 4.2″. (iv.) 4.5″, 2.8″.

67. From the exercises of the last three articles the student has learnt that—

(i.) *In a right-angled triangle the sum of the squares on the sides containing the right angle is equal to the square on the hypotenuse.*

(ii.) *If the square described on one of the sides of a triangle be equal to the sum of the squares described on the other two sides, the angle contained by these two sides is a right angle.*

(iii.) *The middle point of the hypotenuse of a right-angled triangle is equally distant from the three angular points.*

Moreover, it follows from Art. 63 that if one of the three sides of a right-angled triangle and one of the acute angles be given, the triangle can be constructed, for the other two angles are then determinate.

EXERCISES XVI

1 The hypotenuse of a right-angled triangle is 4″, and one of the acute angles is of 60°; construct the triangle, measure its shortest side and the other acute angle.

2. One of the sides containing the right angle of a right-angled triangle is 3″, and the adjacent acute angle is of 40°; construct the triangle.

3. The perpendicular of a right-angled triangle is 7 cm., and the opposite acute angle is of 54°; construct the triangle, and measure the base and hypotenuse correct to the nearest millimetre.

4. Construct a square containing an area of 13 square inches. (Notice that $13=9+4$.)

5. Construct a square containing an area of 11 square inches. (Notice that $11=36-25$.)

6. Construct a square having an area of 2 square inches.

7. Construct squares having areas of 5 and 21 square inches respectively, and find by construction the side of the square whose area is equal to the difference of the areas of these two squares.

The Isosceles Triangle

68. *An* isosceles triangle *has two of its sides equal.*

The unequal side is usually called the base, and the angle opposite to it the vertical angle.

The perpendicular from the vertical angle on the base is the height of the isosceles triangle.

69. *To construct an isosceles triangle, the base and one of the equal sides being given.*

The construction is similar to that of Art. 53. With the extremities of the base as centres, and radii equal to the given sides, we describe arcs which will intersect in the vertex.

EXERCISES XVII

1. On a base 3″ long describe an isosceles triangle having each of the equal sides 2.5 inches in length. Measure the angles at the base.

2. On a base 3.5″ long construct an isosceles triangle, the equal sides being each 5″. Measure the angles at the base. Also draw the bisector of the vertical angle, and measure the segments of the base made by it.

3. Describe an isosceles triangle. the equal sides being each 7 cm. and the base 4 cm. Measure the angles at the base.

Does the bisector of the vertical angle bisect the base also?

4. The equal sides of an isosceles triangle are each 4″ long, and the base 2.4″. Construct it, and measure the angles at the base. Also join the middle point of the base to the vertical angle, and measure the angles which this line makes with the base and with the equal sides. What do you find out?

5. One of the equal sides of an isosceles triangle is 4.8 inches long, and the base is 3.6 inches. Construct the triangle.

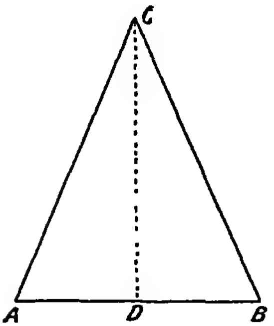
Find the middle point of the base, and through it draw a line at right angles to the base. Does it pass through the vertex?

70. *The angles at the base of an isosceles triangle are equal.*

With any base and length of side construct an isosceles triangle *ABC*. Cut it out in paper. Bring the corners *A* and *B* together, and fold it up double. Then since the sides *AC* and *BC* are equal, the crease must pass through *C*; and let it meet *AB* in *D*.

It will now be seen that the triangles *ADC* and *BDC* are equal to each other in all respects. Hence the angles at *A* and *B* are equal to each other.

We further notice that the line CD divides the vertical angle into two equal parts ACD and BCD, and that it divides the base into two equal parts also.



Again, since the angles ADC and BDC are equal to each other, each of them must be a right angle (Art. 11). Hence we conclude that—

The straight line which bisects the vertical angle of an isosceles triangle bisects the base at right angles.

And that *the straight line which joins the middle point of the base of an isosceles triangle to the vertical angle bisects that angle, and is perpendicular to the base.*

EXERCISES XVIII

1. The base of a triangle is 4″, and the angles at the base are equal, and each is of 30°; construct the triangle, and measure its sides. Are the sides opposite to the equal angles themselves equal?

2. Take a line AB 3.8″ long, and through its middle point D draw DC at right angles to it and 1.9″ long. Join CA, CB. Measure the sides and angles of the triangle ABC. What name would you give this triangle?

3. Take AB 9 cm., and through its middle point D draw DC also 9 cm. Join CA, CB. Measure the angles at the base of this triangle, and also measure the sides opposite to them.

4. Measure AB four inches, bisect it in O, draw OC at right angles to AB, and cut it off one inch. Join AC, BC. Measure the angles at A and B, and the sides opposite to them.

5. Construct the rectangle $ABCD$ with sides 3 and 4 inches long. Draw the diagonals intersecting in O. Describe a circle with centre O and radius OA. Does it pass through B, C, D?

6. Construct the triangle ABC, taking AB 6 cm. in length and each of the angles A and B of 70°. Measure the sides opposite to A and B.

71. From the exercises of the last article we conclude that—

If two angles of a triangle are equal, the sides opposite to them are equal.

Ex. Construct the following isosceles triangles, the bases and vertical angles being given :—

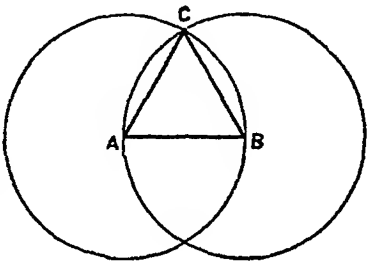
(i.) 8 cm., 54°. (ii.) 3.7″, 72°. (iii.) 3″, 75°. (iv.) 4.6″, 45°.

Equilateral Triangle

72. *An* equilateral triangle *has its three sides equal.*

An equilateral triangle is a particular kind of isosceles triangle, and possesses all the properties of that triangle. Thus, from Art. 70 it follows that *all the angles of an equilateral triangle are equal to one another*, and consequently *each angle of an equilateral triangle is of* 60°.

73. *On a given base AB to construct an equilateral triangle.*



With A and B as centres, and radius AB, describe circles intersecting in C. Then ABC is the required triangle.

EXERCISES XIX

1. On a line AC three inches long construct an equilateral triangle. If B and D be the points of intersection of the two circles used in the construction, show that the figure $ABCD$ is a rhombus.

2. Construct a triangle whose base is 5 cm. and each of the angles at the base of 60°. Measure its sides.

3. Describe a circle of 2″ radius, and at angular intervals of 60° draw radii. Join the extremities of the radii, and examine the triangles thus formed.

4. On a base AB 9 cm long construct an equilateral triangle ABC. Draw the bisectors of the angles at A and B, and let them meet in D. Through D draw DE, DF parallel to CA, CB, meeting AB in E and F. Measure the sides of the triangle DEF and the segments of AB made by E and F.

5. Make an equilateral triangle of one inch side. Through the angular points draw parallels to the sides forming another triangle. Measure the sides of this triangle.

6. The sides of a triangle are $2\frac{1}{8}''$, $1\frac{1}{2}''$, and $2''$. Describe equilateral triangles on the sides externally. Join each angle of the triangle with the opposite vertical angle of an equilateral triangle; measure the three joins. Are they equal?

Numerical Calculations

74. We shall now solve a few numerical examples which depend upon the properties of a right-angled triangle.

Ex. 1. *The sides of a right-angled triangle are 7 ft. 7 in. and 5 ft.; find the hypotenuse*

We have 7 ft. 7 in. = 91 in., and 5 ft. = 60 in.

The square of the hypotenuse $= 91^2 + 60^2$

$= 8281 + 3600$

$= 11,881.$

Therefore the hypotenuse is equal to the square root of 11,881. By the usual process the square root is found to be 109.

Thus the hypotenuse is 109 in., or 9 ft. 1 in.

Ex. 2 *The hypotenuse of a right-angled triangle is* 20, *and the base is* 10; *find the perpendicular.*

Since the sum of the squares of the sides is equal to the square of the hypotenuse, it follows that—

The square of one of the sides is equal to the difference of the squares of the hypotenuse and the other side.

Hence the square of the perpendicular $= 20^2 - 10^2$

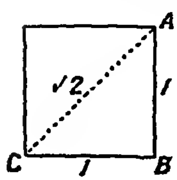
$= 400 - 100$

$= 300.$

Therefore the perpendicular is equal to the square root of 300.

In this case the square root cannot be found exactly, but by proceeding to two places of decimals we get 17.32. Thus the perpendicular is equal to 17.32.

Ex. 3. *To find the diagonal of a square whose side is unity.*



We have

$$AC^2 = AB^2 + BC^2$$
$$= 1^2 + 1^2$$
$$= 2.$$

Therefore $AC = \sqrt{2} = 1.4142$ *nearly*.

Thus we see that when the side of a square is 1, the diagonal is $\sqrt{2}$, or 1.4142.

But all squares are similar, hence we conclude that—

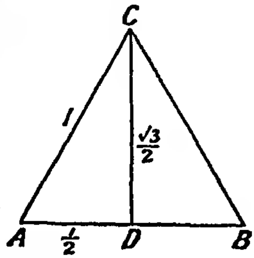
The diagonal of any square is found by multiplying a side by $\sqrt{2}$.

Again, the area of a square, whose side is unity, is evidently 1; and since the diagonal is $\sqrt{2}$, we obtain the following rule for finding the area of a square from the diagonal:—

The area of a square is equal to half the square of a diagonal.

Ex. 4. *To find the height of an equilateral triangle whose side is unity.*

Let D be the middle point of the base, then CD will be perpendicular to AB (Art. 70). In the right-angled triangle ADC the square on CD is equal to the difference of the squares on AC and AD (Ex. 2). That is,



$$CD^2 = 1^2 - (\tfrac{1}{2})^2$$
$$= \tfrac{3}{4}.$$

Therefore CD is equal to $\frac{\sqrt{3}}{2}$.

The square root of 3 is approximately 1.732, hence CD is nearly equal to ·866.

Since all equilateral triangles are similar, we get the following—

RULE.—*The height of an equilateral triangle is found by multiplying the side by* $\frac{\sqrt{3}}{2}$, *or by* ·866.

EXERCISES XX

1. Construct a square whose diagonal is 3 in.

2. The height of an equilateral triangle is 2 in.; construct it.

3. Construct a rhombus, the two diagonals being 2 in. and 3 in. long respectively.

4. Construct a right-angled triangle whose hypotenuse is 5 in. and base 3 in.; and show by actual measurement that the perpendicular is 4 in.

5. On a base 10 cm. long construct an equilateral triangle, and measure its height correct to the nearest millimetre. Is the Rule of Ex. 4, Art. 74, verified?

In the following right-angled triangles the sides are given; find the hypotenuse:—

6. 105 ft., 88 ft.

7. 234 ft., 88 ft.

8. 1056 ft., 2050 ft.

9. 477 ft., 1364 ft.

10. 2 ft. 7 in., 40 ft.
11. 3 ft. 4 in., 3 ft. 6 in.
12. 8 chs. 61 lks., 6 chs. 20 lks.
13. 16 yds. 2 ft., 32 yds. 2 ft. 2 in.

In the following right-angled triangles the hypotenuse and one side are given; find the other side:—

14. 369 ft., 81 ft.
15. 73 yds. 2 ft., 7 yds.
16. 20 yds. 4 in., 2 yds. 4 in.
17. 298 ft., 280 ft.
18. 170 yds., 136 yds.
19. 38 yds 2 ft. 10 in., 14 yds. 1 ft. 4 in.

Find the hypotenuse correct to two decimal places, the sides being—

20. 91 ft., 87 ft.
21. 5 yds. 1 ft., 3 yds.
22. 9 yds. 3 in., 3 yds. 1 ft.
23. 101 ft., 101 ft.

24. Each side of an isosceles triangle is 377 yds, and the base is 690 yds.; find the height.

25. The base of an isosceles triangle is 320 ft., and the height is 78 ft.; find the length of a side.

26. A town *A* is 20 miles west of another town *B*, and 48 miles north of a town *C*; find the distance between *B* and *C*.

27. The sides of a triangle are 6 ft. 1 in. and 4 ft. 4 in., and the height is 4 ft.; find the base.

28. Find correct to the nearest inch the diagonal of a square, of which the side is 10 ft.

29. The side of an equilateral triangle is 96 yds.; find the height correct to the nearest inch.

30. The height of an equilateral triangle is 8 ft.; find the length of a side.

31. Each side of a rhombus is 4 ft., and one of the diagonals is also 4 ft.; find the length of the other diagonal.

32. Find in acres the area of a square field, of which the diagonal is 440 yds.

33. The area of a square field is one-tenth of an acre; find its diagonal.

34. The diagonal of a rectangular field is 244 yds., and one of the sides is 44 yds; find the area in acres, etc.

35. The area of a square field is 2 acres, 46,402 sq. links; find the diagonal.

36. Find the length of a ladder which, standing at a distance of 7 ft. from a wall 24 ft. high, will just reach the top.

37. A string 68 ft. long is stretched from the top of a tower, and reaches the opposite edge of a moat 32 ft. wide running round the base of the tower; find the height of the tower.

38. A ladder 100 ft. long stands upright against the wall of a

building. How many inches will the top descend if the foot of the ladder be pulled away 10 ft. from the wall?

39. A ladder 25 ft. in length, when placed in a street, reaches a window 24 ft. from the ground on one side, and being turned over it reaches another window 20 ft. high on the other side. Find the breadth of the street.

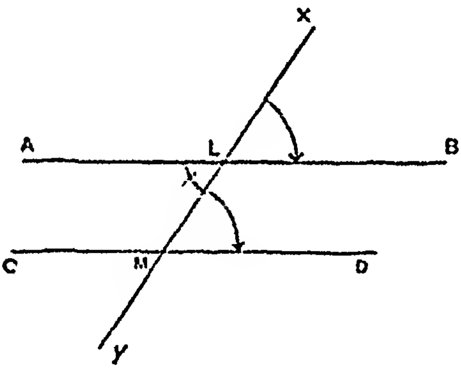
40. The sides of a rectangle are 90 ft. and 120 ft. respectively; the diagonal joining two opposite corners is drawn and divided in the ratio of 1 : 2. Find the distances of the point of division from the two other corners of the rectangle.

CHAPTER VII

PARALLELOGRAM

75. Draw two parallel lines AB, CD, and draw a line XY cutting them both in the points L and M.

Measure the angles marked with arrow heads. You

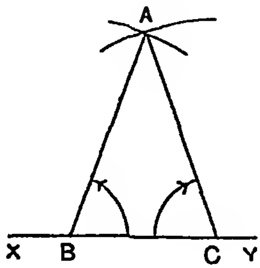


will find that each contains the same number of degrees. You may also test their equality by cutting out the upper angle and fitting it to the lower.

By repeating this experiment in a variety of cases we conclude that—

Parallel lines make equal angles, measured in the same sense, with any straight line.

In the figure above, both angles are measured from XY in a *clock-wise sense.* In the isosceles triangle ABC, the sides AC, AB make equal angles with XY, but while the angle at C is measured in a clock-wise sense, the angle at B is measured in a counter clock-wise sense from XY, and the lines AB, AC are not parallel.

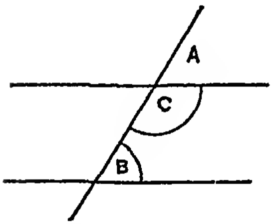


A pair of angles, such as XLB and LMD, one exterior and the other interior, on the same side of the line XY, are called **corresponding angles.**

A pair of interior angles, such as ALM and DML, on opposite sides of XY, are called **alternate angles.**

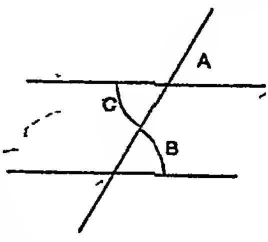
76. From the equality of corresponding angles, which we have established above, certain other conclusions can be drawn without further measurement and testing.

The angles A and C are together equal to two right angles (Art. 45).



But A is equal to B. Therefore the angles B and C are together equal to two right angles. Hence

If a straight line cut two parallel straight lines it makes the interior angles on the same side together equal to two right angles.



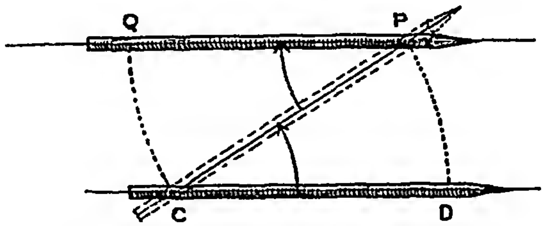
Again, the vertically opposite angles A and C are equal.

But A is equal to the corresponding angle B. Therefore the angles B and C are equal. Hence

If a straight line cut two parallel straight lines, it makes alternate angles equal.

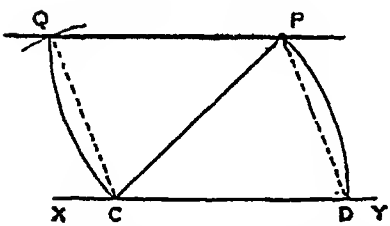
Remembering that parallel lines are such as are drawn in the same direction (Art. 14), we may illustrate the last proposition in the following manner :—

Place a pencil along the line *CD*, and rotate it counter-clock-



wise into the dotted position passing through the point *P*. The original direction of the pencil was *CD*, and its direction has been altered by turning it round *C* through an angle. In order to give it the same direction as before, in a different position, we must turn it round *P* through an equal angle in a contrary direction, *i.e.* in a clock-wise direction.

77. *Through a given point P to draw a straight line parallel to a given straight line XY.*



In *XY* take any point *C*. With centre *C*, and radius *CP*, describe an arc cutting *XY* in *D*.

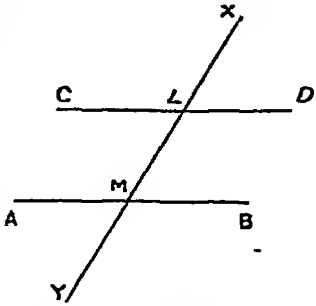
With centre *P*, and radius *PC*, describe an arc, and with centre *C*, and radius equal to *DP*, describe another arc cutting it in *Q*. Join *PQ*. Then *PQ* will be parallel to *CD*.

For the three sides of the triangle *PCD* are equal respectively to the three sides of the triangle *PCQ*; hence the triangles are congruent.

Therefore the alternate angles *PCD*, *QPC* are equal.

EXERCISES XXI

1. Draw two straight lines an inch apart, and another straight line cutting them as in the figure. You have four exterior angles and four interior angles; point them out. Also point out the pairs that form corresponding angles, and the pairs that form alternate angles.



2. With your protractor measure all the angles in the figure of the last exercise, and note them down in their places. Now look at your figure, and explain how the measurement of a single angle would have been sufficient to determine the rest.

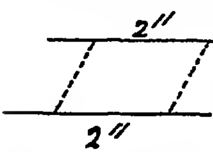
3. Take a line AB two inches long, and through A and B draw straight lines, making angles of 60° with AB: (i.) on the same side of AB in the same sense; (ii.) on the same side of AB in opposite senses; (iii.) on opposite sides of AB in the same sense. See which of these pairs are parallel.

4. Give reasons why straight lines perpendicular to the same straight line are parallel.

5. Take a piece of paper with a straight edge, and fold it so that the edge falls exactly on itself: fold it again so that the doubled edge falls on itself; and so on. Now unfold. Are the creases parallel? Why?

6. Draw two parallel lines, and a third line cutting them both. Make cuts along these lines, and fit together the different angles to verify the propositions of Articles 75 and 76.

7. Draw two parallel lines, and mark on each two points at a distance of 2 inches from one another. Join the extremities of these lines as shown in the figure. Measure these lines and the angles which they make with one of the parallel lines. What do you discover?



Repeat this experiment with other parallel lines and different equal distances marked on them to show that—

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are themselves equal and parallel.

8. On a base 2″ describe an isosceles triangle, having each of its equal sides 3.2″. Produce one of the equal sides through the vertex, and

bisect the exterior angle thus formed. Is the bisector parallel to the base of the triangle? Give reasons.

9. Take AB 3 4" long, and draw AD, BC, making angles of 30° with AB, and each of length 2 2 inches. Join DC. Are the lines DC, AB parallel? Measure their distance.

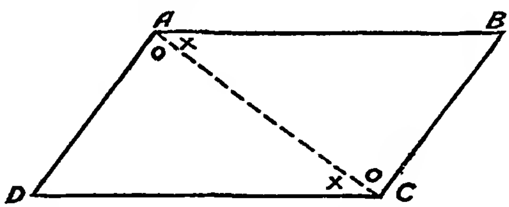
10. Draw two lines making an angle of 40°; take a point anywhere, and through it draw two lines parallel to the former, each to each; measure the angle between the last pair of lines.

Make the same experiment with pairs of lines inclined at different angles. What do you find out?

Parallelogram

78. In Art. 23 we have defined a parallelogram, and shown how it may be constructed when two adjacent sides and the contained angle are given.

Draw a pair of parallels, and another pair of parallels



to cut both of them; you will then obtain a parallelogram $ABCD$. Cut it out, and make a cut across the diagonal AC. Fit the pieces together by placing opposite sides of the parallelogram on each other. You will find that the two triangles are exactly equal to one another, so that $AB = CD$, $BC = AD$, the angle at B = the angle at D. Also, since the marked angles are equal, the angle at A = the angle at C.

Try this with other parallelograms and the result will be the same; we therefore conclude that—

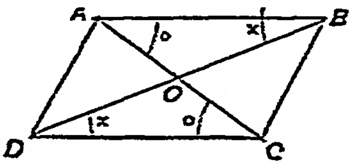
The opposite sides and angles of a parallelogram are equal, and each diagonal bisects the parallelogram.

We might have arrived at this result without any experimenting, for in the two triangles ABC, ADC the side AC is common, and two angles in the one are equal to two angles in the other, X, X and O, O being alternate angles; therefore the triangles are congruent (Art. 56).

79. *The diagonals of a parallelogram bisect one another.*

This may be shown by cutting along the diagonals and fitting together the pieces, or else we may argue thus :—

In the two triangles OAB, OCD the marked angles are equal, being alternate angles, and the side AB is equal to the side CD. Therefore the triangles are congruent, and have those sides equal which are opposite to the equal angles. Thus OA equals OC, and OB equals OD.



EXERCISES XXII

1. The diagonal of a parallelogram is 5" long, and the sides are 3.5" and 4" long respectively; construct it.

2. The diagonal of a parallelogram is 4.4" long, and the sides make with it angles of 35° and 54° respectively; construct it, and measure the other diagonal.

3. Draw two lines AOC and AOD, intersecting in O at an angle of 30°. Cut off $OA = OC = 2.7"$, and $OB = OD = 3"$. Is the quadrilateral $ABCD$ a parallelogram? Give reasons.

4. Take a line of small length, and show how to bisect it by means of the two set-squares only.

5. On a base 3" long construct a triangle with sides 3.4" and 2.9" long. Join the middle point of the base to the middle points of the sides. Measure the sides of the quadrilateral figure formed. Is it a parallelogram?

6. One of the sides of a rhombus is 2" long, and one of the angles

is of 50°. Construct the rhombus, and cut it out; show, by folding, that the diagonals bisect the angles of the rhombus.

Can you prove this by general reasoning?

7. Take two parallel lines, and cut them off of equal length; prove that the straight lines which join their extremities cross-wise bisect one another.

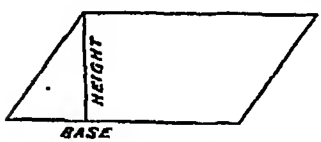
8. Make two equal triangles, having sides 2″, $2\frac{1}{2}$″, and 3″ long; cut them out and fit them together, so as to form a parallelogram.

In how many different ways can you do this?

9. On a base 4″ long construct a triangle with sides 3.2″ and 3.7″ long. Through the angular points draw parallels to the opposite sides Show by measurement that the sides of the triangle so formed are each twice as long as the corresponding sides of the original triangle.

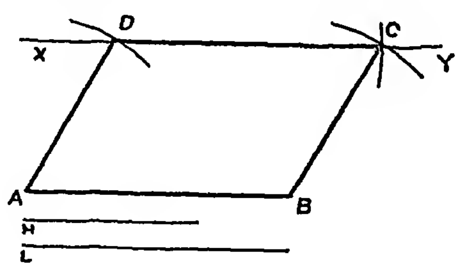
Can you give reasons for this fact?

80. Any side of a parallelogram may be chosen as **the base**, and then the perpendicular distance between it and the opposite parallel side is called **the height**,



On a given base AB to construct a parallelogram, having a given height H, and a given side L, not less than the height.

Draw XY parallel to AB at a distance equal to H



(Art. 20). With centre A, and radius L, describe an arc cutting XY in D. Cut off DC equal to AB, and join BC.

Then $ABCD$ is the required parallelogram.

EXERCISES XXIII

1. On a line 3.5″ long construct a parallelogram whose height is 1.8″, and the length of a side is 2.3″.

2. In the second figure of Art. 80, if you complete the circle whose centre is A and radius L, it will cut the line XY in two points, such as D; show that you can obtain two parallelograms satisfying the given conditions.

3. On a line 2.5″ long construct a parallelogram whose height and length of a side are both 1.5″.

Point out any peculiarity in the construction, and discover whether the parallelogram constructed is of a special kind.

4. In the second figure of Art. 80, take AB 3.7″ long, and H and L 2″ and 1.7″ long respectively. What happens? Can you now explain why the "side must not be less than the height"?

5. Take a line 4″ long, and on it construct a parallelogram, such that the distance between this line and the opposite side is 2″, and one of the angles of the parallelogram is of 60°.

6. On a line 2.6″ long construct a parallelogram having a height of 1.4″ and a diagonal of 3.2″.

7. The diagonals of a parallelogram are 2″ and 4″ respectively, and the angle contained by them is of 60°. Construct the parallelogram, and measure its angles.

8. Construct a parallelogram whose diagonals are 6″ and 8″, and one of whose sides is 5″. Examine the figure, and find out what it is called.

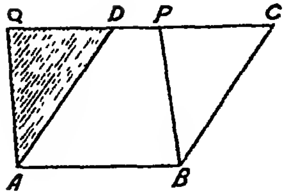
9. The adjacent sides of a parallelogram are equal, each being 2.2″ long, and the diagonal makes angles of 30° with either side. Construct the parallelogram, and measure the length of the other diagonal.

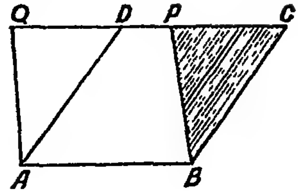
10. Construct a parallelogram whose diagonals are 2.8″ and 3.6″, and whose height is 1.4″.

81. Construct two parallelograms $ABCD$, $ABPQ$ on the same base AB and between the same parallels AB and QC.

Make another figure which is an exact copy of the first, and cut out both. If the drawings are properly made, the figures will be equal to one another.

From the first figure cut off the triangle AQD, and from



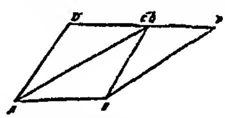


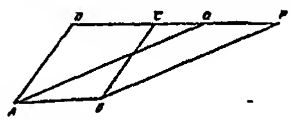
the second the triangle *BPC*. By applying these triangles to one another you will find that they are equal; hence the remaining portions, *i.e.* the areas *ABCD* and *ABPQ*, are equal.

By repeating this experiment with several pairs of parallelograms we conclude that—

Parallelograms on the same base and between the same parallels are equal in area.

It may happen in some cases that the point *Q* coincides with *C*, or it may lie in *DC* produced, but in all cases the method of proceeding is the same, viz. make two figures exactly alike, and from one of them remove the triangle *BPC*, and from the other the triangle *AQD*. The portions removed are seen to be equal, and hence we infer that the remaining parallelograms *ABCD*, *ABPQ* are equal.





The student will note that the parallelograms are equal in area, but not equal in other respects.

From two figures which are equal in all respects we remove portions which are also equal in all respects, but we do not remove them from the same *positions*; hence the remaining areas are equal, but they have not the same shape.

It is easy to see that the triangles *BPC*, *ADQ* of the first figure are equal without having recourse to experiment. For in the two triangles the sides *AD* and *BC* are equal,

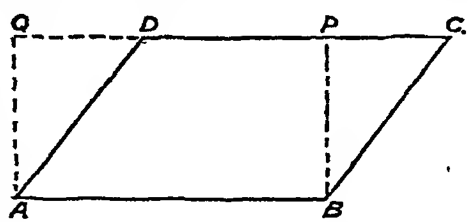
being opposite sides of a parallelogram, and a little consideration will show that the angles at their bases are also equal. Hence the triangles are equal in all respects (Art. 56). We further notice that the parallelogram $ABCD =$ (whole figure) − triangle BPC, and the parallelogram $ABPQ$ = (whole figure) − triangle ADQ; therefore the parallelogram $ABCD =$ the parallelogram $ABPQ$.

82. Since parallel lines are at the same perpendicular distance from each other throughout their lengths, we can enunciate the proposition of the last article thus :—

Parallelograms on the same base and having the same height are equal to one another.

In particular, since a rectangle is a parallelogram, we have—

A rectangle and a parallelogram having the same base and height are equal in area.



We can now find the area of any parallelogram $ABCD$. For we can construct the equivalent rectangle $ABPQ$, having the same base and height as the parallelogram, and the area of this rectangle is known when AB and AQ are known.

Hence the area of a parallelogram may be found by the following—

Rule.—*Multiply the base by the height.*

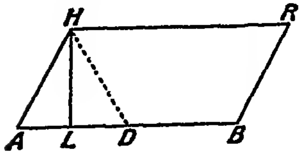
Ex 1. *The base of a parallelogram is 5 ft. 6 in., and the height is 4 ft.; find the area.*

The area $= (5 \text{ ft. } 6 \text{ in.}) \times 4$ ft.

$$= \frac{11}{2} \times 4 = 22 \text{ sq. ft.}$$

Ex. 2. *The two sides of a parallelogram are 3 ft. and 6 ft., and the included angle is of 60°; find the area.*

Let $AH = 3$ *ft.*, $AB = 6$ *ft*, and the angle $HAB = 60°$. Draw HL perpendicular to AB, then HL will be the height. Take LD equal to LA and join HD. The triangles HLA and HLD are equal, therefore the angle HDL is of 60°. Hence the triangle HAD is equilateral.



Now the height of an equilateral triangle is equal to $\frac{\sqrt{3}}{2}$ times the side (Ex. 4, Art. 74); therefore $HL = 3 \times \frac{\sqrt{3}}{2}$.

The area of the parallelogram $= HL \times AB$

$$= 3 \times \frac{\sqrt{3}}{2} \times 6$$
$$= 9\sqrt{3} = 9 \times 1.732$$
$$= 15.588 \text{ sq. ft.}$$

Note.—When the angle between the given sides is of 30°, 60°, 45°, 150°, 120°, or 135°, the height can easily be found by means of a suitable construction. In this connection the diagrams of Exs. 3 and 4 of Art. 74 will be found useful.

EXERCISES XXIV

1. The sides of a parallelogram are 3 in. and 5 in. respectively, and the included angle is of 45°; construct it.

2. The diagonals of a parallelogram are 3 in. and 4 in. respectively, and the angle contained by them is of 30°; construct the parallelogram.

In the following parallelograms the bases and heights are given; find the areas:—

3. Base 20 ft. 3 in., height 12 ft.

4. Base 6 yds. 2 ft., height 2 yds. 4 in.

5. Base 11 chs. 16 lks., height 3 chs. 72 lks.

6. Base 1 fur. 2 yds., height 23 yds.

7. Base 26 yds. 2 ft. 10 in., height 17 yds. 1 ft. 8 in.

8. The area of a parallelogram is 722 sq. yds. 2 sq. ft., and the base is 43 yds. 1 ft.; find the height.

9. The area of a parallelogram is 1 acre, and the height is 29 yds. 1 ft.; find the base.

10. The two sides of a parallelogram are 2 ft. 3 in. and 1 ft. 3 in., and the lesser height is 10 in. Find the greater height.

11. In the parallelogram $ABCD$, of which the area is 1 acre, the perpendiculars from D on BC and AB are 27 yds. 1 ft. 6 in. and 44 yds. respectively; find the sides.

In the following parallelograms the lengths of the two unequal sides and the angle contained by them are given; find the areas:—

12. Sides 5 ft., 12 ft.; contained angle 30°.

13. Sides 3 yds., 2 yds.; included angle 45°.

14. Sides 5 in., 4 in.; included angle 60°.

15. Sides 1 ft., 3 ft.; included angle 120°.

16. Sides 100 yds., 50 yds.; contained angle 135°.

17. Sides 33 chains, 22 chains; contained angle 150°.

18. Each side of a rhombus is 50 ft., and one of the diagonals also is 50 ft.; find the area.

19. The area of a rhombus is one-half the area of a square which has the same perimeter; determine the angles of the rhombus.

20. The sides of a rectangle are 3 cm. and 5 cm.; construct a parallelogram equal in area to the rectangle, and having one of its angles of 50°.

21. Construct a parallelogram equal in area to a square of 5 cm. side, and having one of its angles of 60°.

22. Construct a parallelogram having an area of 40 square centimetres, a perimeter of 36 centimetres, and a base of 10 centimetres. Measure its angles.

23. Construct the square $ABCD$, having a side of 2.8 inches; bisect AB, CD, in P and Q respectively. Find the area of the parallelogram $APCQ$, and show by actual measurement that the lines AQ, PC trisect the diagonal BD.

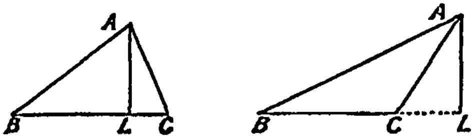
24. Take a line BD three inches in length, and on it, as a diagonal, construct a parallelogram $ABCD$. Bisect AB, CD in P and Q, join AQ, PC, and show that these lines trisect BD.

Also prove in any manner that the parallelogram $APCQ$ has half the area of the parallelogram $ABCD$.

CHAPTER VIII

AREA OF A TRIANGLE

83. ANY side of a triangle may be chosen as its **base**, and then the perpendicular on it from the opposite angle is called its **height** or **altitude**.



Thus, if we take BC as the base of the triangle ABC, the perpendicular AL is the height.

Thus, every triangle has three altitudes, or perpendiculars, and we can easily show that all these pass through the same point.

For, fold the triangle ABC so that BC falls on itself while the crease passes through A. We thus mark the perpendicular AL. In the same way the other two perpendiculars can be marked, and if the folding is done carefully, it will be found that all three pass through the same point.

This point is called the **orthocentre** of the triangle.

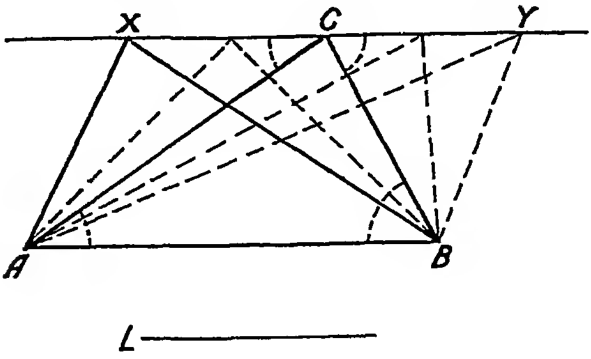
Let the student draw five triangles having the following sides:—

(i) 13 cm., 14 cm., 15 cm. (ii.) $2\frac{1}{2}''$, $3\frac{1}{2}''$, $5''$. (iii.) 6 cm., 6 cm., 6 cm. (iv.) $3.4''$, $2.8''$, $3.4''$. (v.) $3''$, $4''$, $5''$.

Find the orthocentres of these triangles by using the construction of Art. 62. Note that when the triangle is obtuse-angled, as in (ii.), the orthocentre falls without the triangle, and when the triangle is right-angled, as in (v.), the orthocentre coincides with an angular point.

84. *To construct a triangle having a given base AB and a given altitude L.*

Draw XY parallel *to* AB, and at a distance L from it.



Take any point C in XY, and join AC, BC; then ABC is a triangle constructed on the base AB, and having an altitude equal to L.

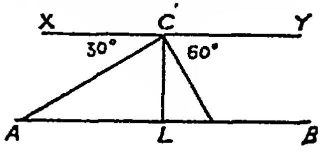
Since all points on the line XY are at a distance L from AB, it is evident that by taking C in different positions on the line XY we can make any number of triangles satisfying the given conditions.

The student will notice that *when the base and altitude of a triangle are given, the vertex lies on a straight line parallel to the base at a known distance from it, and that the sides make angles with this line which are equal to the base angles of the triangle* (Art. 76).

EXERCISES XXV

1. The height of a triangle is 3″, and the angles at the base are of 30° and 60°; construct it.

Draw CL 3″ in length. Through C and L draw XCY, ALB at right angles to CL. Make the angles XCA, YCB of 30° and 60° respectively.



2. Construct a triangle whose height is 7 cm., and the angles at the base of 40° and 60° respectively.

3. Construct an isosceles triangle whose height is 2.8″, and each of the angles at the base is of 36°.

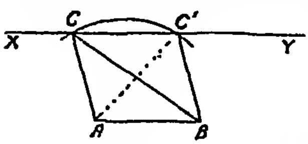
4. Construct an isosceles triangle whose height is 8.6 cm., and the vertical angle is of 36°.

The vertical angle being given, the angles at the base are known.

5. Construct an equilateral triangle of four inches height.

6. Construct an equilateral triangle whose height is 17.3 cm., and measure one of its sides to the nearest millimetre.

7. Construct an isosceles triangle whose height is 7 cm., and whose vertical angle is a right angle. Measure the base, and also measure one of the equal sides to the nearest millimetre.



8. The base, height, and one side of a triangle being given, construct it.

Draw XY parallel to the base AB at a distance equal to the height. With centre A, and radius equal to the given side, describe an arc cutting XY in C, C′. Then CAB, C′AB are two triangles satisfying the given conditions. As in Art. 80, there may be only one triangle in some cases, and in others none.

9. On a base 8 cm. long construct a triangle whose altitude is 5 cm., and one of the sides is: (i.) 6.5 cm.; (ii.) 5 cm.; or (iii.) 4.5 cm.

In case (ii.) point out any peculiarity in the triangle.

10. Construct an isosceles triangle whose base and altitude are 6″ and 4″ respectively.

In this case, through the middle point of the base draw a perpendicular equal to the height.

11. Construct a triangle having a given height and sides equal to two given lines.

At one extremity of the given height draw a straight line at right angles to it, and with the other extremity as centre describe arcs whose radii are equal to the lengths of the given sides. These arcs will in general cut the base in four points. You will thus get two triangles satisfying the given conditions.

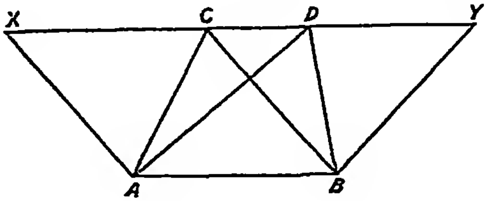
12. Construct a triangle whose altitude is 2.7″, and sides are 3.3″ and 4″ respectively.

13. Construct a triangle whose altitude is 5 cm., and sides 8 cm. and 5 cm. respectively.

14. Construct an isosceles triangle whose height is 3″, and each of the equal sides 4.4″.

15. Construct an equilateral triangle 10 cm. in height; find its orthocentre, and find by measurement the ratio of the parts into which the orthocentre divides each of the perpendiculars of the triangle.

85. *Triangles on the same base and between the same parallels are equal in area.*



Let ABC, ABD be two triangles on the same base AB, and between the same parallels AB, XY.

Draw BY, AX parallel to AD and BC respectively.

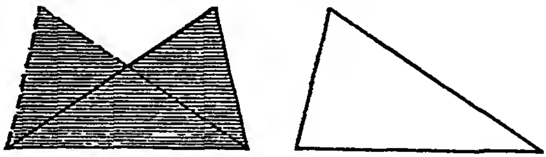
The figures $ABCX$, $ABYD$ are parallelograms on the same base AB, and between the same parallels AB, XY; therefore they are equal (Art. 81).

But the triangles ABC, ABD are halves of these parallelograms (Art. 78); therefore they are also equal.

Since parallel lines are at the same distance from each other throughout their lengths, we may express this proposition thus:—

Triangles on the same base, and having the same altitude, are equal in area.

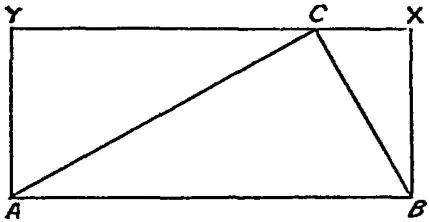
Again, since two triangles having equal bases can be so



placed as to have a common base, we conclude that—

Triangles having equal bases and equal altitudes are equal in area.

We have seen that the triangle ABC is half of the parallelogram $ABCX$, therefore it is equal in area to half of any other parallelogram having the same base and height.



In particular, the area of a triangle is half the area of a rectangle having the same base and height.

But the area of a rectangle is equal to the product of the base into the height. Hence *the area of a triangle* $= \frac{1}{2}$ (*base* × *height*).

Ex. 1. *The base of a triangle is* 8 *ft., and the height is* 5 *ft.* 6 *in.: find the area.*

The area

$$= \tfrac{1}{2}(8 \times \tfrac{11}{2}) \text{ sq. ft.}$$
$$= 22 \text{ sq. ft.}$$

Ex. 2. *The area of a triangle is* 84 *sq. ft., and the base is* 14 *ft.; find the height.*

We have

$$base \times height = 2(area);$$

therefore

$$height = \frac{2(area)}{base}, \text{ and } base = \frac{2(area)}{height}.$$

Here the height

$$= \frac{2 \times 84}{14} = 12 \text{ ft.}$$

86. When the three sides of a triangle are given, the area can be found by the following—

Rule.—*From half the sum of the three sides subtract each side in succession, multiply together the half sum and the*

three remainders; the square root of the product is the area of the triangle.

The general proof of this rule is too difficult for insertion here, but the student can easily verify its correctness in the particular case of any right-angled triangle.

Ex. 1. *The sides of a triangle are* 13, 14, *and* 15; *find the area.*

Half the sum of the sides

$$=\tfrac{1}{2}(13+14+15)=21;$$

and

$$21-13=8,\ 21-14=7,\ 21-15=6.$$

Hence the area

$$\begin{aligned}&=\sqrt{21\times8\times7\times6}\\&=\sqrt{7\times3\times4\times2\times7\times3\times2}\\&=7\times4\times3\\&=84.\end{aligned}$$

Ex. 2. *The sides of a triangle are* 7, 24, *and* 25; *find the area.*

Half the sum of the sides$=28$; and $28-7=21$, $28-24=4$, $28-25=3$.

Hence the area

$$=\sqrt{28\times21\times4\times3}=84.$$

Again, since

$$7^2+24^2=49+576=625=25^2,$$

the triangle is right-angled; therefore, if we take 7 as the base, the perpendicular height is 24, and the area

$$=\tfrac{1}{2}\times7\times24=84.$$

Thus we see that in this particular case the rule gives the area correctly.

EXERCISES XXVI

1. The base of a triangle is 17 yds. 2 ft. 9 in., and the height is 9 yds. 1 ft.; find the area in square yards, etc.
2. The base of a triangle is 5 poles, and the height is 5 yds; find the area in square yards.
3. The base of a triangle is 5629 links, and the height is 625 links; find the area in acres.

4. The sides of a right-angled triangle are 280 and 102 ft.; find the area.

5. The hypotenuse of a right-angled triangle is 234 ft., and one of the sides is 90 ft.; find the area.

6. Each side of an equilateral triangle is 12 chains; find the area in acres and square links.

7. The sides of a triangular field are in the ratio of 3 : 4 : 5, and the perimeter is 1 mile; find the price that must be paid for it at £7 : 10s. per acre.

8. The base of a triangle is 72 ft., the height 25 ft., and one of the sides is 48 ft.; find the other side.

9. Each side of an isosceles triangle is 100 ft., and the perpendicular height is 60 ft.; find the area.

In the following triangles the sides are given; find the areas:—

10. 18, 41, 41.
11. 39, 42, 45.
12. 26, 40, 42
13. 20, 37, 51.
14. 20, 34, 42.
15. 25, 113, 132.

Find to two decimal places the areas of the triangles whose sides are—

16. 5, 6, 7.
17. 9, 9, 11.
18. 19, 21, 22.
19. 47, 53, 56.

20. The sides of a triangle are 26 yds. 1 ft. 9 in., 37 yds., 37 yds. 2 ft. 9 in.; find the area.

21. The sides of a triangle are 4, 5, and 7 ft.; find the area correct to the nearest square inch.

22. The sides of a triangle are 5, 6, 7; prove that the area is $6\sqrt{6}$.

23. The sides of a triangle are 7, 8, 9; show that the area is $12\sqrt{5}$.

24. The sides of a triangle are 3, 5, 7; show that its area is to that of an equilateral triangle having the same perimeter as 3 : 5.

25. Two sides of a triangular field containing an obtuse angle are 110 and 220 yds. Find the length of the third side, that the field may contain exactly an acre.

26. The diagonal of a rectangle is 2 ft., and one of the sides is 1 ft. Compare its area with that of an equilateral triangle, described on the lesser side

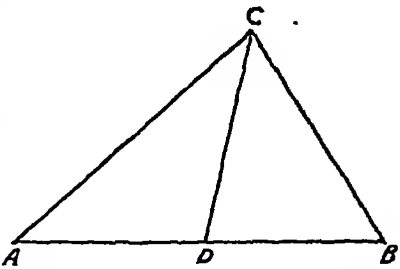
27. The sides of a triangle are 13, 14, 15; find the lengths of the perpendiculars from the angular points on the opposite sides.

28. An equilateral triangle and a square have the same perimeter; compare their areas.

87. We shall now give a few constructions which depend upon the propositions of this chapter.

To bisect a triangle ABC by a straight line drawn through one of the angular points C.

Bisect the base AB in D. Join CD. The two triangles CAD, CBD have equal bases and the same height, therefore they are equal (Art. 85). Hence CD divides the triangle ABC into two triangles of equal area.



In the same way, by dividing the base AB into any number of equal parts, and joining the points of division to the vertex, we can divide the triangle into a number of equal triangles.

To construct a parallelogram equal in area to the triangle ABC, and having one of its angles equal to a given angle.

Through C draw XY parallel to AB. At D make an angle equal to the given angle, and let the containing line meet XY in R. Then BD, RD are the sides of the parallelogram.

To construct an isosceles triangle equal in area to the triangle ABC, and standing on the same base AB.

Through D draw DS at right angles to AB to meet XY in S. Then SAB is the required triangle.

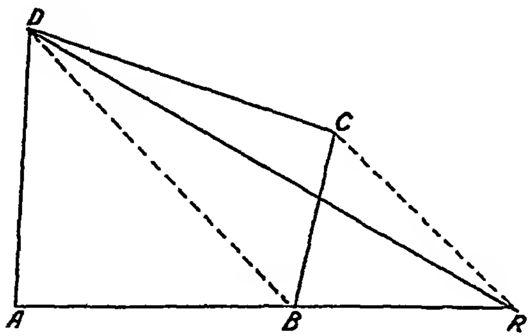
To construct a triangle equal in area to the triangle ABC, and having one of its angles equal to a given angle.

Make the given angle at the point A, and let the containing line meet XY in T. Then TAB is constructed as required.

88. *To construct a triangle equal in area to a given quadrilateral ABCD.*

Join BD. Through C draw CR parallel to BD to meet AB produced in R. Join DR.

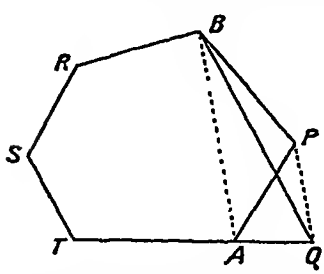
Then *DAR* is the required triangle. For the triangle *DAR* and the quadrilateral *ABCD* have the portion



ABD common to both, and the parts *DBR*, *DBC* are also equal, since they are triangles on the same base *DB*, and between the same parallels *DB* and *CR*.

Therefore the triangle *DAR* is equal to the quadrilateral *ABCD*.

89. *To reduce an irregular rectilineal figure to an equivalent triangle.*

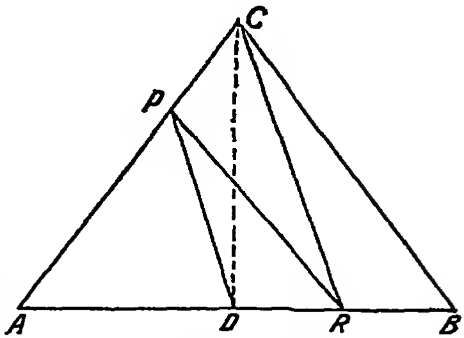


Let *APBRST* be an irregular figure. Join *AB*; and through *P* draw *PQ* parallel to *AB*, meeting *TA* produced in *Q*. Join *BQ*.

The triangles *APB*, *AQB* are equal (Art. 85). Therefore the given six-sided figure *APBRST* is equivalent to the five-sided figure *TQBRS*.

By repeating this construction the given figure will finally be reduced to a triangle.

90. *To bisect a given triangle ABC by a line drawn through a given point P in one of its sides.*



Bisect AB in D. Join PD. Draw CR parallel to PD. Join PR.

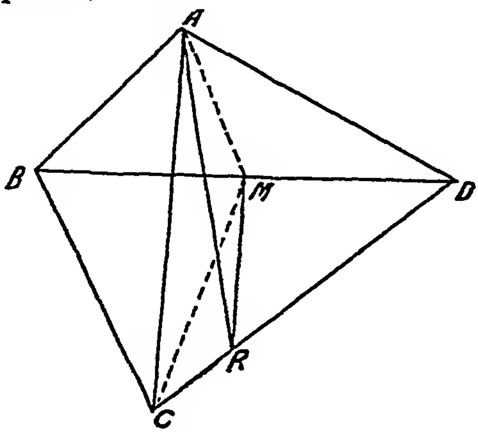
Then PR will bisect the triangle.

The triangles PDR, PDC are equal (Art. 85). Add to each of these the triangle PDA, then the triangles APR and ACD are equal.

But ACD is half of ABC.

Therefore APR is half of ABC.

To bisect the quadrilateral $ABCD$ by a line drawn through the angular point A.



Bisect BD in M, and draw MR parallel to AC. Then AR will bisect the quadrilateral.

Notice that the quadrilateral $ABMC$ is half of the quadrilateral $ABCD$. The remainder of the proof is not difficult.

MISCELLANEOUS EXERCISES

1. Draw six concentric circles with radii $1''$, $1.25''$, $2''$, $2.25''$, $3''$, $3.25''$.

2. Describe two circles with radii $3''$ and $4''$ and centres $5''$ apart.

3. Describe two circles each passing through the centre of the other.

4. Draw two circles with radii 5 cm. and 8 cm. and centres 13 cm. apart.

5. Draw two circles with radii $1.5''$ and $2.8''$ and centres $1.3''$ apart.

6. Draw a straight line 10 cm. in length; measure it in inches, and thence find the number of inches in a metre.

7. Divide a straight line 21 cm. long into four equal parts.

8. Draw any three lines meeting at a point, and measure the three angles formed by them. Is the sum of the angles equal to four right angles?

9. Draw a line AB $6''$ long; through its middle point draw any two lines lying on the same side of it, and measure the three angles so formed. Is their sum equal to two right angles?

10. Draw two intersecting lines, and measure all the angles formed by them. Are the angles equal in pairs? What is the sum of two unequal angles? What is the sum of all the angles?

11. Construct a triangle with sides $2\frac{3}{4}''$, $3''$, $3\frac{1}{4}''$ and measure all its angles. Is the sum of the angles equal to two right angles?

12. Draw an equilateral triangle of $3''$ side, and show by measurement that all its angles are equal. How many degrees are there in one of its angles?

13. On a base $2.5''$ long describe an isosceles triangle having each of its equal sides $3.5''$ long. Measure the angles of this triangle and find their sum.

14. Draw a triangle with sides 4 cm., 7 cm., 9 cm., measure all its angles and find their sum.

15. Construct a triangle with sides $3''$, $4''$, $4\frac{1}{2}''$ and produce its longest side; show by measurement that the exterior angle so formed is equal to the sum of the two interior opposite angles.

16. Repeat the last exercise taking a triangle whose sides are 8 cm., 11 cm., and 13 cm. long.

17. Draw the triangle ABC, having given $BC=3''$, $AB=2.8''$ and $\angle B=75^\circ$. Produce the sides in order (see Art. 50) and measure all the exterior angles; their sum ought to be equal to four right angles.

18. Repeat the last exercise with a triangle ABC, in which $BC=4''$, $\angle B=30^\circ$, $\angle C=60^\circ$.

19. Taking a base AB $4''$ long describe above it a triangle whose other sides are $3''$ and $3\frac{1}{2}''$, and describe below it another triangle whose sides are $2''$ and $4''$. You thus obtain a quadrilateral whose diagonal is AB.

Measure all the angles of this quadrilateral and find their sum.

Also produce the sides in order and find the sum of all the exterior angles.

Are the two sums equal to one another?

20. Draw any five-sided figure, and find by measurement the sum of all its interior angles which ought to be equal to six right angles.

Produce the sides in order and verify that the sum of all the exterior angles is equal to four right angles.

21. Draw any large six-sided figure. Measure all the interior angles, and verify that their sum is equal to eight right angles.

22. Construct a triangle with sides $2.8''$, $3''$, and $3.4''$. Join the middle point of each side to the opposite vertex and notice that the three joins pass through the same point.

23. Repeat the experiment of the last exercise taking a triangle whose sides are 11 cm., 13 cm., and 15 cm. long.

24. Draw the triangle ABC, having given $AB=4''$, $CA=3.5''$ and $\angle A=60^\circ$. From the angular points draw perpendiculars on the opposite sides and notice that these perpendiculars pass through the same point.

25. Repeat the last experiment taking a triangle ABC, in which $AB=8$ cm., $BC=10$ cm. and $\angle B=60^\circ$.

26. Construct the triangle ABC, in which $BC=75$ mm., $\angle B=45^\circ$, $\angle C=60^\circ$.

Draw the three perpendiculars of the triangle as in the last two exercises.

27. Draw the triangle ABC, having given $AB=3.7''$, $\angle A=43^\circ$, $\angle B=47^\circ$.

Draw the three perpendiculars. Where do they meet?

28. Draw the triangle ABC, in which $BC=9$ cm., $\angle A=45^\circ$, $\angle B=30^\circ$.

Show that the three perpendiculars of the triangle meet outside the triangle.

29. Construct a triangle ABC, in which $AB=3.2''$, $AC=2''$ and $\angle B=32^\circ$.

Show that two triangles can be found with the given parts.

30. Draw the triangle ABC, having given $AC=11$ cm., $AB=5.5$ cm. and $\angle C=30°$.

Show that the triangle is right-angled.

31. Draw an angle of 73° and copy it by construction.

Check your construction by measurement with the protractor.

32. Use the protractor to make an angle of 43° and by construction obtain an angle twice as large.

33. Make a triangle with sides 2.8″, 3.1″, 3.8″ and draw the bisectors of its three angles.

These bisectors ought to pass through the same point.

Show that the distances of this point from the sides of the triangle are equal.

34. Repeat the last experiment with a triangle whose sides are 10.2 cm., 11.3 cm., and 12.2 cm.

35. Draw a triangle with sides 13 cm., 14 cm., 15 cm.; bisect two of its angles, and let the bisecting lines meet in O; with O as centre and a radius 4 cm. long describe a circle.

What is peculiar about this circle?

36. Draw again the triangle of the last exercise, and produce the two smaller sides; bisect the exterior angles so formed by lines which meet in O. With O as centre and a radius 14 cm. long describe a circle.

37. Take a line 4.8″ long, and draw a straight line bisecting it at right angles.

38. Draw a straight line 3.5″ long, and draw its right bisector cutting it off 1.75″ both above and below the bisected line; join the extremities of the two lines. What figure do you obtain?

39. Take a line 4″ long, and draw its right bisector cutting it off 2″ above and 3.5″ below; join the extremities of the two lines. The figure formed is called a kite.

40. Construct the triangle ABC, having given $BC=3.6''$, $\angle B=57°$, $\angle C=68°$; draw the right bisectors of its three sides, and note that they pass through the same point.

This point is equidistant from A, B, and C, hence you are enabled to describe a circle passing through the angular points of the triangle.

41. Draw a triangle with sides 2.9″, 3.2″, 3.5″ and describe the circle which passes through its angular points.

42. Draw an equilateral triangle of 7 cm. side and describe the circle passing through its angular points.

43. Construct a right-angled triangle in which the sides containing the right angle are 1.5″ and 3.6″ respectively. Find by measurement the length of the hypotenuse, and verify the theorem of Art. 66.

44. Construct a square containing an area of 29 square inches.

(*N.B.* $29=25+4$.)

45. Construct a square containing an area of 5 square inches.

46. The hypotenuse of a right-angled triangle is 12 cm. and one of the sides is 7 cm.; construct the triangle.

47. Draw the triangle ABC, in which $AB=4''$, $CA=3.2''$ and $\angle C=90°$.

48. Describe a square containing an area of 20 square centimetres.

(*N.B.* $20=36-16$.)

49. A ladder 25 feet long is placed against a wall, with its foot 8 feet from the bottom of the wall, and fails to reach a window, but when it is drawn a foot nearer to the wall it just reaches the window; show that the height of the window is 24 feet.

50. On a base 8 cm. long construct an isosceles triangle having each of the equal sides 12 cm. in length.

Measure the angles at its base.

51. On a base $2.9''$ long construct an isosceles triangle having each of the angles at the base of 54°.

Measure the sides to see that they are equal.

52. Describe an isosceles triangle with sides 7 cm., 11.5 cm., 11.5 cm.; bisect the angle between the equal sides, and notice that the bisector passes through the middle point of the third side.

53. On a base $2''$ long describe an isosceles triangle whose perimeter is $8''$; join the middle point of the base to the vertex, and show that this line is perpendicular to the base and bisects the vertical angle.

54. Describe an isosceles triangle whose vertical angle is of 54° and whose base is $4.3''$ long.

(First calculate the base angles.)

55. On a base 7.5 cm. long describe an equilateral triangle whose vertical angle is of 36°.

56. Construct an equilateral triangle whose perimeter is 20 cm.

57. Draw an equilateral triangle which has the same perimeter as a square of $1.5''$ side.

58. The base of an isosceles triangle is 2.8 inches, and the vertical angle is of 30°; construct an equilateral triangle having the same perimeter.

59. Draw an isosceles triangle of $4''$ height on a base $3''$ long.

60. On both sides of a base 7 cm. long describe isosceles triangles whose equal sides are 10 cm.

What is this four-sided figure called?

61. On both sides of a base $2.6''$ long describe equilateral triangles,

and show that the diagonals of the quadrilateral so formed bisect each other at right angles.

62. Draw two parallel lines 2″ apart, and draw a third line, cutting them both and making an angle of 60° with one of them; write down the values of all the angles in the figure so formed.

63. Take a line *AB* 4″ long, and draw a parallel to it at a distance of 2.5″; anywhere on this parallel cut off *DC* 4″ in length; join *AD*, *BC* and verify that the figure *ABCD* is a parallelogram.

64. Construct a parallelogram whose sides are 3″ and 4.6″ and the included angle is of 60°.

Show that the diagonals bisect each other.

65 Describe a parallelogram whose base is 8 cm., height 4.5 cm., and one of its angles is of 60°.

66. Draw a parallelogram with sides 4″ and 5″, and having a height of 3″; cut it across so that the two pieces when fitted together may form a rectangle.

67. Draw a triangle *ABC*, having $AB = 12$ cm., $BC = 10.5$ cm., $AC = 8$ cm., and describe a parallelogram having its base coincident with *BC*, its area equal to the area of *ABC*, and one of its angles of 60°. (Art. 87.)

68. Construct a triangle with sides 2.7″, 3″, 3.2″ and draw a rectangle equal to it in area.

69. Draw a triangle *ABC*, in which $BC = 9$ cm., $\angle B = 44°$, $\angle C = 65°$, and construct an isosceles triangle equal to it in area and standing on the side *BC* as base. (Art. 87.)

70. Draw a triangle *ABC*, having given $BC = 3″$, $AB = 3.8″$, $\angle B = 54°$, and construct an isosceles triangle of equal area standing on *AB* as base.

71. Draw a triangle *ABC* with sides 2.4″, 2.7″, 3.1″ and construct on its largest side as base a triangle of equal area having one of its angles of 60°.

72. Draw any five-sided figure, and reduce it to a triangle of equivalent area.

73. Construct a triangle *ABC*, having given $AB = 9.4$ cm., $BC = 10.5$ cm., $CA = 11$ cm., and from a point in the longest side, distant 3 cm. from *C*, draw a straight line bisecting the triangle.

BOOK III

CHAPTER IX

SIMILAR FIGURES

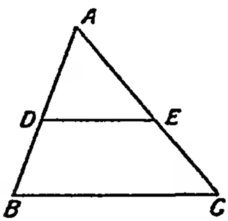
91. *If a straight line be drawn parallel to one side of a triangle, it cuts the other sides proportionally.*

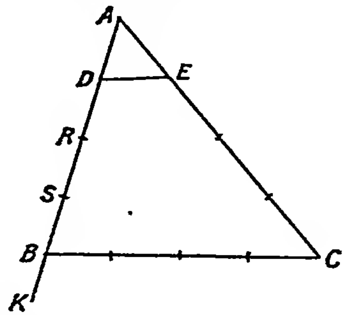
Draw DE parallel to BC; then

$$AD : DB :: AE : EC.$$

Thus AD is the same fraction of DB as AE is of EC. Or, we may say, that AD is the same fraction of AB as AE is of AC.

Take any line AK, and on it step off the distances AD, DR, RS, SB, all equal to each other. Draw a line AC,



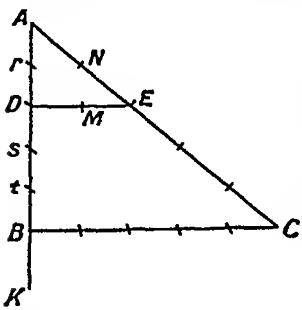


making any angle with AB, and having any convenient length. Join BC. Through D draw DE parallel to the base BC of the triangle ABC.

If the figure has been correctly drawn you will find that the length AE is contained exactly three times in EC. Thus when AD is one-third of DB, it is found that AE is one-third of EC.

Moreover, you will find that DE is contained exactly four times in BC.

Thus each side of the triangle ADE is one-fourth of the corresponding side of the triangle ABC.



For another illustration draw a line AK, and on it step off the equal distances, Ar, rD, Ds, st, and tB. Draw the line AC, and join BC. Draw DE parallel to the base BC of the triangle ABC. Bisect DE and AE in the points M and N.

You will find that AN is contained exactly three times in EC, and that DM is contained exactly five times in BC. Hence

$$AD : DB :: AE : EC :: 2 : 3.$$

Moreover, we see that each side of the triangle ADE is two-fifths of the corresponding side of the triangle ABC.

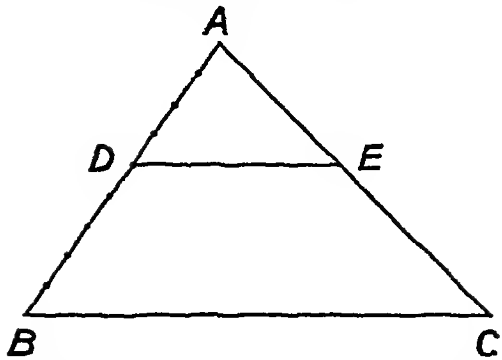
92. *Conversely, if the sides of a triangle are divided in the same ratio the straight line joining the points of division is parallel to the base.*

Draw any triangle ABC, and divide each of the sides AB, AC in the ratio of $4 : 5$ (Art. 21), so that

$$\frac{AD}{DB} = \frac{4}{5}, \text{ and } \frac{AE}{EC} = \frac{4}{5}.$$

Join DE.

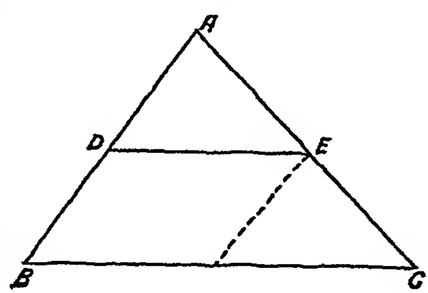
It will be found that DE is parallel to BC.



The same experiment may be tried with other triangles and other ratios; in all cases the line joining the points of division will be found parallel to the base.

93. The following particular cases of the propositions of the last two articles are deserving of notice :—

If from the middle point of a side of a triangle a straight line is drawn parallel to the base, it bisects the other side, and is equal to one half of the base.



The line joining the middle points of two sides of a triangle is parallel to the base, and is equal to one half of the base.

EXERCISES XXVII

With the following data construct the first figure of Art. 91, and calculate the lengths of the lines AE, EC, and DE :—

1. $AB=3''$, $AC=4''$, $BC=5''$, $AD/DB=2/3$.
2. $AB=4.5''$, $AC=5''$, $BC=6.5''$, $AD/DB=1/5$.
3. $AB=6.5$ cm., $AC=7$ cm., $BC=7.5$ cm., $AD/DB=5/8$.
4. $AB=11$ cm., $AC=5$ cm., $BC=12$ cm., $AD/DB=4/7$.

5. $AB=3''$, $AC=3.6''$, $BC=4.2''$, $AD/AB=5/6$.
6. $AB=5''$, $AC=5.5''$, $BC=6''$, $AD/AB=3/7$.
7. $AB=9$ cm., $AC=7$ cm., $BC=10$ cm., $AB:BD=9:5$.
8. Draw any triangle ABC, and in the sides AB, AC find the points P and Q such that $AP/PB=3/4$, and $AQ/QC=3/4$; join PQ, and show that it is parallel to BC.

Also measure BC and PQ, and find what fraction PQ is of BC.

9. Draw a triangle whose sides are $4''$, $5''$, and $6''$ long. Find the middle points of the sides, and join them to form a new triangle. What are the lengths of the sides of this triangle?

Also measure the angles, and find out if they are equal to the angles of the original triangle. Can you give reasons why they should be equal?

10. Construct an isosceles triangle whose base is $4''$, and each of the equal sides $5''$. From the middle point of the base draw straight lines parallel to the sides and meeting them, and find their lengths. What sort of a parallelogram do they form with the sides?

11. Take BD 4 inches long, and with it, as diagonal, construct a quadrilateral figure $ABCD$ such that $AB=3''$, $BC=3.5''$, $CD=4''$, and $DA=2.5''$. Join the middle points of the sides to form another quadrilateral, and prove that it is a parallelogram whose sides are half as long as the diagonals of $ABCD$.

12. The base of a triangle is $4''$, and the sides are $3''$ and $3\frac{3}{4}''$ respectively. From a point in the side $3''$, distant $2''$ from the vertex, a line is drawn parallel to the base. Find the length of this line, and the lengths of the segments into which it divides the other side.

The Rule of Three

94. The proposition of Art. 91 may be used to find, by geometrical construction, a fourth proportional to three given numbers.

Ex. 1. Find a fourth proportional to 2, 3, 4. Draw OA $2''$ long, and produce it. Cut off AB $3''$.

Draw OX, making any angle with OA, and cut off OC $4''$. Join AC. Through B draw BD parallel to AC, meeting OX in D. Then the number of inches in the length of CD is the required fourth proportional. For, since AC is parallel to BD, we have $\dfrac{OA}{AB}=\dfrac{OC}{CD}$

Ex. 2. Find a fourth proportional to the numbers 23, 32, 41. Here it will be more convenient to represent unity by the tenth of an inch, so that in the construction of the last exercise we take OA, AB, and OC of lengths 2.3″, 3.2″, and 4.1″ respectively.

In some cases a millimetre may be employed to represent unity.

Ex. 3. If 5 yards of cloth cost Rs.2, 13a., find by geometrical construction the price of 7 yards.

Ex. 4. Find a fourth proportional to the numbers 13, 19, 23.

Ex. 5. A field took 24 men 16 days to reap; find by geometrical construction how long would it take 72 men.

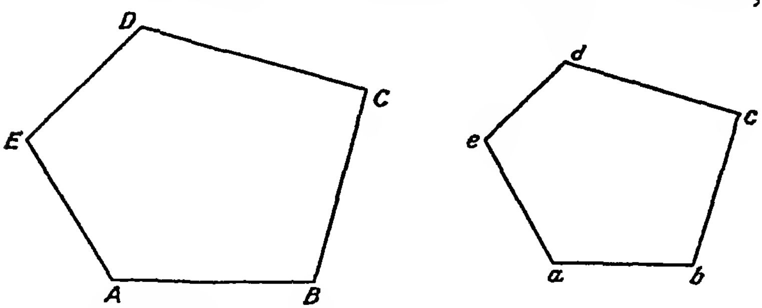
Ex. 6. In seven hours a clock loses twenty-five seconds, find approximately the number of seconds it will lose in a day.

Ex. 7. The sum of Rs.100 is to be divided between two persons in the ratio of 7 : 12. Find by geometrical construction their approximate shares.

95. In ordinary language two figures are said to be similar to each other when they have the same shape but not the same size. Thus a large equilateral triangle and a smaller equilateral triangle are similar figures; squares described on two lines of different lengths are similar figures; and so on. For purposes of calculation, however, we require a more exact—

DEFINITION.—**Similar rectilineal figures** *are equiangular to one another, and their sides about the equal angles are proportional.*

Let the two figures $ABCDE$ and $abcde$ be similar;



then the angles A, B, C, D, E taken in order are equal

respectively to the angles a, b, c, d, e taken in order; and $\frac{AB}{ab}=\frac{BC}{bc}=\frac{CD}{cd}=\frac{DE}{de}=\frac{EA}{ea}$. Thus if the side AB be double of ab, then every side of the first polygon will be double of the corresponding side of the second polygon.

Again, if $AB : ab :: 3 : 2$; then

$BC : bc :: 3 : 2$, and so on.

We see then that *similar figures must have* (1) *their angles equal, and* (2) *the ratios of their corresponding sides, which lie between pairs of equal angles, also equal.*

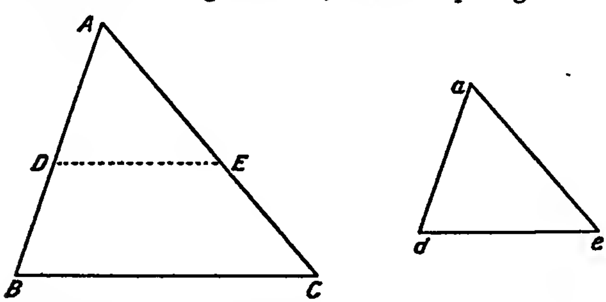
In the next article we shall show that in the case of a triangle when one condition of similarity is satisfied the other follows.

Exercise.—In the similar figures given above, $AB=32$, $BC=32$, $CD=40$, $DE=26$, $EA=27$ feet; and $cd=30$ feet. Find the remaining sides of the smaller figure.

Ans. $ab=24$, $bc=24$, $de=19\frac{1}{2}$, $ea=20\frac{1}{4}$ feet.

96. *Equiangular triangles are similar.*

Let the two triangles ABC, ade be equiangular to one



another, having the angles A, B, C equal to the angles a, d, e respectively. Then shall

$$\frac{AB}{ad}=\frac{BC}{de}=\frac{CA}{ea}.$$

Cut out the triangle ade, and place it on ABC so that

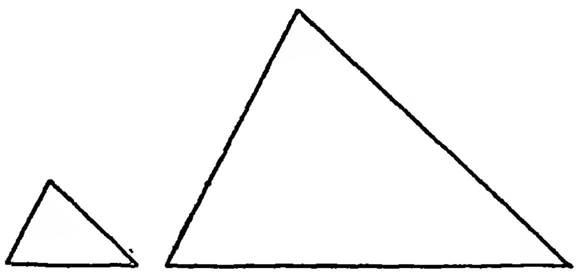
the point a may coincide with A; then since the angles A and a are equal, the side ad will lie on AB, and ae on AC. Let DE be the new position of de. Now the angles ADE and ABC being equal, the line DE is parallel to the base BC of the triangle ABC (Art. 75). But a straight line drawn parallel to the base of a triangle cuts off from it another triangle whose sides are proportional to those of the original triangle (Art. 91).

Hence $$\frac{AD}{AB}=\frac{DE}{BC}=\frac{EA}{CA};$$

or, $$\frac{ad}{AB}=\frac{de}{BC}=\frac{ea}{CA}.$$

Hence triangles are similar when their angles are equal.

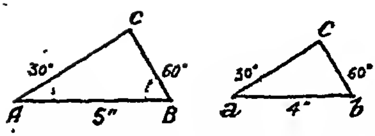
Again draw any triangle, and then draw another whose sides are three times as great as the corresponding sides of



the first. It will be found on measurement that the angles of the two triangles are equal.

Similarly construct other pairs of triangles such that the sides of the one are the same multiples of the sides of the other; on measuring angles it will be found that each pair is equiangular. Hence *if the sides of two triangles are proportional, the triangles are equiangular, and therefore similar.*

EXERCISES XXVIII



1. Take two lines 5" and 4" long respectively, and on them construct triangles, each of which has angles at the base of 30° and 60°. Measure the sides of these triangles, and show from your measurements that $\frac{AC}{ac}=\frac{BC}{bc}=\frac{AB}{ab}=\frac{5}{4}$.

Note that in forming a ratio such as $\frac{AC}{ac}$ *we take the sides of the two triangles which are opposite to equal angles. Such sides are called* corresponding *sides.*

2. As in the last example take bases of 8 cm. and 4 cm., and on them construct triangles whose base angles are of 36° and 45°. Show that the sides of one triangle are each twice as long as the corresponding sides of the other.

3. Take BC and DE two and three inches long respectively; construct the triangles ABC, DEF such that $\angle B = \angle D = 75°$, and $\angle C = \angle E = 45°$. Prove by actual measurement that the corresponding sides of the triangles ABC, DEF are in the ratio of 2 : 3.

4. On bases 3.9 cm. and 6.5 cm. long construct isosceles triangles whose vertical angles are of 30°. Measure the sides of these triangles, and show that they are similar, the corresponding sides being in the ratio of 3 : 5.

5. Construct a triangle whose sides are 2", 2.4", and 3" long, and another whose sides are each twice as long. Prove by measuring the angles that the triangles are equiangular.

6. Make a triangle whose sides are 3.6", 4.2", and 4.8" long, and another whose sides are 2.4", 2.8", and 3.2" long; cut them out, and by fitting together the angles show that they are equiangular.

7. Construct a triangle whose sides are 3.2", 4", and 5.2", and another whose sides are each five-eighths of the sides of the first. Cut out the smaller triangle, and by placing it on the larger show that they are equiangular.

8. Construct the right-angled triangle whose sides are 3", 4", and 5"; draw a perpendicular from the right angle on the hypotenuse. Make a duplicate of this figure, and cut out the two small right-angled triangles into which the large triangle is divided by the perpendicular. By placing each of these triangles on the first figure and on

each other, prove that all three are equiangular, and consequently similar.

9. Repeat the experiment of the last exercise with the right-angled triangle whose sides are 5 cm., 12 cm., and 13 cm., and also with any other right-angled triangle drawn at random.

State in general terms the conclusion you draw from these experiments.

10. Draw a triangle whose sides are 9 cm., 10 cm., and 13 cm., and on a base 7 cm. construct a triangle similar to it.

How many such triangles can you construct? Make a separate figure for each one of them.

11. Draw the following pairs of triangles:—

(i) $AB=3''$, $AC=4''$, $\angle A=60°$;
$A'B'=4\frac{1}{2}''$, $A'C'=6''$, $\angle A'=60°$.

(ii) $AB=7$ cm., $AC=8$ cm., $\angle A=54°$;
$A'B'=10.5$ cm., $A'C'=12$ cm., $\angle A'=54°$.

(iii) $AB=2''$. $AC=3''$. $\angle A=45°$;
$A'B'=4''$, $A'C'=6''$, $\angle A'=45°$.

Measure the angles and see that the two triangles in each pair are equiangular, and hence similar. Also notice that in each case the sides about the given equal angles are proportional. thus $3:4::4\frac{1}{2}:6$.

From these experiments we conclude that:—

If two triangles have one angle of the one equal to one angle of the other and the sides about these equal angles proportional, the triangles are similar.

12. Draw a triangle ABC, having given $AB=1''$, $BC=1\frac{1}{2}''$, $CA=1\frac{1}{4}''$ and construct another triangle $A'B'C'$, in which the angle A'=angle A, but the sides containing A' are twice as long as the given sides containing A. Show by measurement of angles that the two triangles are equiangular.

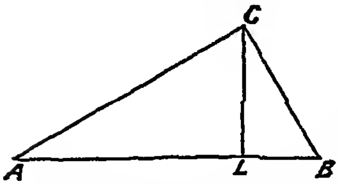
Right-angled Triangle

97. We shall now give a formal proof of the results arrived at in the experiments of Exs. 8 and 9 of the last article.

In a right-angled triangle, if a perpendicular be drawn from the right angle to the hypotenuse, the two triangles so formed are similar to each other and to the original triangle.

Let CL be the perpendicular.

Each of the angles CAL and BCL is the complement of the angle ACL; hence they are equal. Again, each of the angles CBL and ACL is the complement of the angle BCL; therefore they are equal. Thus the triangles ACL, BCL are equiangular, and hence they are similar (Art. 96).



Since the triangles ACL and BCL are similar, their sides are proportional; so that we have

$$AL : CL :: CL : BL.$$

Now when four quantities are proportional, the product of the means is equal to the product of the extremes; hence

$$CL^2 = AL \times BL.$$

That is,

$$(\textit{perpendicular})^2 = \textit{product of segments of base.}$$

Next consider the triangles ALC, ABC. The angle at A is common to both, and one angle of each is a right angle; therefore the triangles are equiangular, and hence similar. We have

$$AC : AL :: AB : AC;$$

hence $$AC^2 = AB \times AL \qquad \text{(i.)}$$

and similarly $$BC^2 = AB \times BL \qquad \text{(ii.)}$$

That is,

$$(\textit{a side})^2 = \textit{base} \times \textit{adjacent segment.}$$

By adding (i.) and (ii.) we get

$$\begin{aligned} AC^2 + BC^2 &= AB \times AL + AB \times BL \\ &= AB \times (AL + BL), \end{aligned}$$

or $$AC^2 + BC^2 = AB^2.$$

That is,

sum of the squares of sides = square of hypotenuse.

EXERCISES XXIX

1. The sides of a right-angled triangle are 3″ and 4″. Find the length of the perpendicular from the right angle on the hypotenuse; and find also the lengths of the segments of the hypotenuse made by the perpendicular.

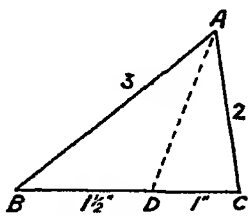
2. The hypotenuse of a right-angled triangle is $6\frac{1}{2}''$, and one of the sides is 6″. Construct the triangle, and find the segments of the base made by a perpendicular from the right angle on the hypotenuse.

3. Construct a right-angled triangle whose hypotenuse is $6\frac{1}{4}''$ and one side $1\frac{3}{4}''$. Find the segments of the hypotenuse made by a perpendicular from the right angle on the hypotenuse.

4. ABC is an equilateral triangle of 3″ side. AD is drawn perpendicular to BC, and DE perpendicular to AB. Find the lengths of BE and AE.

Bisector of the Vertical Angle

98. On a base $2\frac{1}{2}''$ construct a triangle ABC whose sides are 2″ and 3″. Draw the bisector of the vertical angle, meeting the base in D. You will find on measurement that $BD = 1\frac{1}{2}''$, and $CD = 1''$; and consequently $BD:DC = 3:2$. Thus in this case the segments of the base made by the bisector of the vertical angle are in the ratio of the sides.



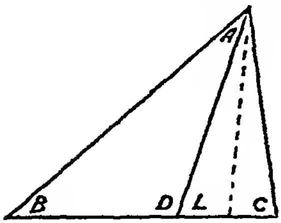
Repeat the same experiment with other triangles, and you will find that in every case—

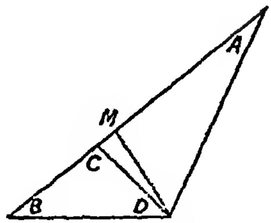
The bisector of an angle of a triangle divides the opposite side in the ratio of the sides containing the angle.

Draw a triangle ABC, and draw AD bisecting the

vertical angle and AL perpendicular to the base. Then

$$\frac{\text{Triangle } ABD}{\text{Triangle } ACD}=\frac{\frac{1}{2}\,AL\times BD}{\frac{1}{2}\,AL\times CD}=\frac{BD}{CD}.$$





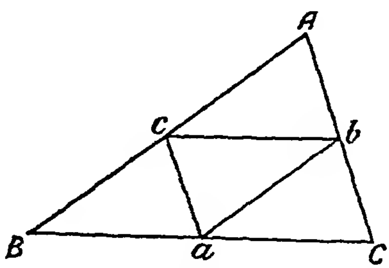
Now cut out the triangle and fold it about AD. In this position draw DM perpendicular on AB. Then

$$\frac{\text{Triangle } ABD}{\text{Triangle } ACD}=\frac{\frac{1}{2}\,DM.AB}{\frac{1}{2}\,DM.AC}=\frac{AB}{AC}.$$

$$\text{Therefore } \frac{BD}{CD}=\frac{AB}{AC}; \text{ i.e. } BD:CD=AB:AC.$$

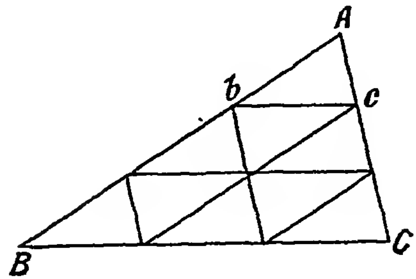
Ex. Construct a triangle whose sides are 4", 5", and 6". Draw the bisectors of the angles, and calculate the lengths of the segments which each makes on the side it meets.

99. Take a triangle and join the middle points of its



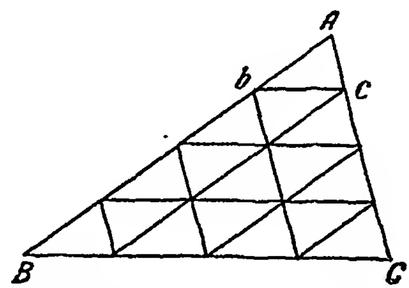
sides. We thus obtain four equal triangles similar to one another and to the original. Thus $\frac{\text{area } Abc}{\text{area } ABC}=\frac{1}{4}$.

Again, trisect the sides of a triangle and join the points



of trisection as in the figure. The triangle *Abc* is similar to the triangle *ABC* (Art. 96), and all the nine small triangles are equal. Hence

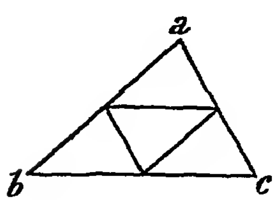
$$\frac{\text{area } Abc}{\text{area } ABC}=\frac{1}{9}.$$

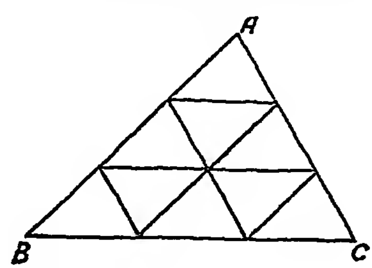


Similarly in the third figure

$$\frac{\text{area } Abc}{\text{area } ABC}=\frac{1}{16}.$$

Lastly, construct two similar triangles on bases of 2″ and 3″, and divide them up into smaller triangles as in Figures





1 and 2. The smaller triangles in both figures are equal to one another; hence $\frac{\text{area } abc}{\text{area } ABC}=\frac{4}{9}.$

From the above experiments we conclude that—

The areas of similar triangles are to one another as the squares of their corresponding sides.

EXERCISES XXX

1. The sides of a triangle are 3″, 4″, and 5″; find its area, and hence write down the area of a triangle whose sides are 6″, 8″, and 10″ respectively.

2. Find the area of a triangle whose sides are 13 ft., 14 ft., and 15 ft., and from the result obtain the area of a triangle whose sides are 6 ft. 6 in., 7 ft., and 7 ft. 6 in.

3. On bases 4″ and 5″ construct two triangles, each having its angles at the base of 30° and 60°. Divide the first triangle into 16 equal triangles, as in the third figure of Art. 99, and divide the second triangle into 25 equal triangles; hence show that the areas of the triangles are in the ratio of $4^2 : 5^2$.

4. Construct two squares whose sides are $1\frac{3}{4}$″ and $3\frac{1}{2}$″, and show by means of a diagram that the larger square is four times as great as the smaller.

5. The areas of two similar triangles are in the ratio of 49 : 121, and the sides of the smaller triangle are 2.1″, 2.8″, and 3.5″; find the sides of the larger.

6. The sides of one triangle are 3″, $3\frac{1}{4}$″, $3\frac{3}{4}$″, and of another 4″, $4\frac{1}{3}$″, 5″; compare their areas.

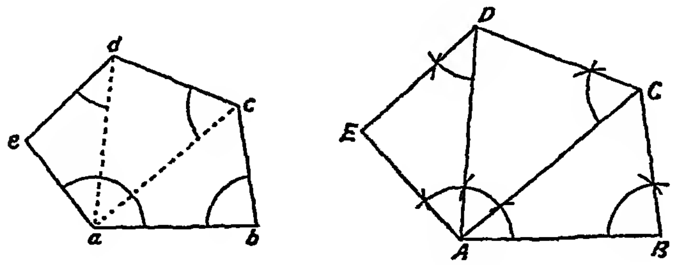
7. On a base 4″ long construct a triangle whose height is 3″ and one of the sides is 4″; at a distance of $1\frac{1}{2}$″ from the base draw a parallel to it. Compare the areas of the two triangles thus formed.

8. In the triangle of the last example parallels to the base are drawn at distances of 1″ and 2″ from it respectively. Compare the areas of the three triangles in the figure, and find the area of the strip between the two parallels and the sides of the triangle.

100. *On a given straight line AB to construct a rectilineal figure similar to a given rectilineal figure abcde.*

From one of the angular points of the given figure draw straight lines to the other angular points, dividing it into a

number of triangles. Make the angles CAB, CBA equal to the angles cab, cba. We thus get a triangle ABC similar to



the triangle abc. Again, make the angles DAC, DCA equal to the angles dac, dca; and the angles EDA, EAD equal to the angles eda, ead.

Both the figures are now divided into the same number of similar triangles.

It is an easy exercise for the student to prove that the two figures are equiangular, and that their sides are proportional.

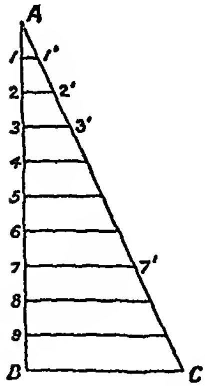
Exercise 1. Draw any four-sided figure, and make another similar figure whose sides are half as long.

Exercise 2. Draw any six-sided figure, and construct another similar figure whose sides are to the sides of the original figure as $2:3$.

101. **Diagonal Scales.**—It is often found convenient to have a scale on which we can read off three dimensions as units, tenths, and hundredths, or yards, feet, and inches. In this case some of the subdivisions on a plain scale become very minute, and hence diagonal scales are used.

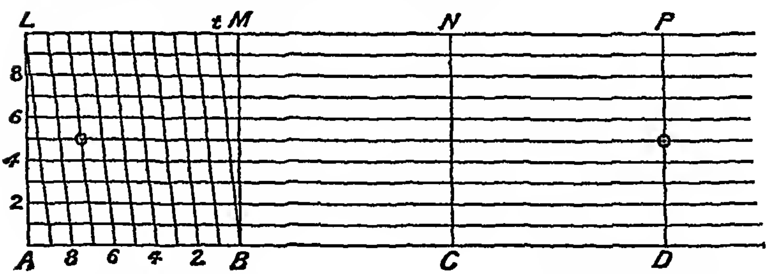
The principle on which all such scales are constructed can be easily understood by considering the triangle ABC, in which the side AB is divided into a number of equal

parts, say 10, and lines are drawn from the points of division parallel to the base BC. All the triangles in the figure, which have a common vertex A, are similar to the triangle ABC. Hence, since $A1$ is $\frac{1}{10}$ of AB, therefore $11'$ is $\frac{1}{10}$ of BC. Similarly, $22'$ is $\frac{2}{10}$ of BC, $33'$ is $\frac{3}{10}$ of BC, and $77'$ is $\frac{7}{10}$ of BC. Thus the parallels through 1, 2, 3, etc., are respectively $\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \ldots \frac{9}{10}$ of BC (Art. 91). We have now found the several multiples of the tenth part of BC, and it is evident that the method is of practical use even when the line BC is very small. We shall apply this method to draw a **decimal diagonal scale.**



102. *To construct a diagonal scale to show inches, tenths, and hundredths.*

Draw a line 6 in. long and divide it into 6 equal parts, AB, BC, CD, etc., each 1 in. in length.



Through the points of division draw perpendiculars AL, BM, CN, etc.; and make AL 1 in.

Divide AB, AL, LM each into 10 equal parts. Through the points of division of AL draw parallels to AD, and join the points of division of AB and LM by oblique lines as

shown in the figure. The scale is now complete, though in the figure, for want of space, only a part of it is shown. Suppose we require to take off 2.7 in.; then place one point of the compasses at *D*, and stretch out the other to the division marked 7 in *AB*. The distance between the two points will be 2.7 in.

Again, suppose we require 2.75 in.; then place the feet of the compasses on the points marked in the figure by small circles. Notice that here we have two entire inches between the lines *PD* and *MB*; there are also 7 lengths, each equal to one-tenth of an inch; and lastly, the portion of the line between the sides of the triangle *BMt* is five-tenths of *tM*, that is, five-hundredths of an inch. Thus the whole length of the line between the small circles is 2.75 in.

We may also remark here that if *AB* represent 10 ft., then the distance measured above will be 27.5 ft., and if *AB* represent 100 ft., then the same distance will be 275 ft.; and so on.

Exercise 1. Construct a diagonal scale 1 in. to the yard, to read yards, feet, and inches.

[Divide *AB* into 3, and *AL* into 12, equal parts.]

Exercise 2. Construct a diagonal scale, $\frac{3}{4}''$ to $1'$, to show inches and eighths of an inch.

Exercise 3. Draw a diagonal scale, representative fraction $\frac{1}{528}$, to read chains and links, and make it long enough to measure 4 chains.

[100 links = 1 chain = 66 feet. Take $AB = AL = 1\frac{1}{2}''$; and divide each into 10 equal parts.]

CHAPTER X

IRREGULAR RECTILINEAL FIGURES

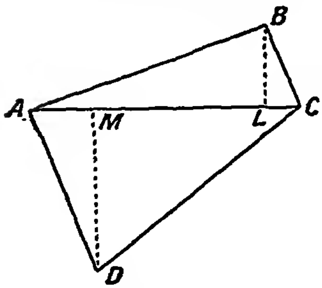
Quadrilaterals

103. THE area of an irregular quadrilateral can be found by dividing it into two triangles.

Thus if one diagonal and the perpendiculars upon it from the outlying angles be known, we can apply Art. 85; and if all the sides and one diagonal be known, the rule of Art. 86 will give the area.

EX. 1. *The diagonal of a trapezium is* 5 *ft., and the perpendiculars upon it from the outlying angles are* 2 *ft. and* 3 *ft.; find the area of the trapezium.*

Let BL and DM be the perpendiculars on the diagonal AC.



The area of the triangle ABC

$=\frac{1}{2}BL \times AC,$

and the area of the triangle ADC

$=\frac{1}{2}DM \times AC;$

therefore the *area of the quadrilateral*

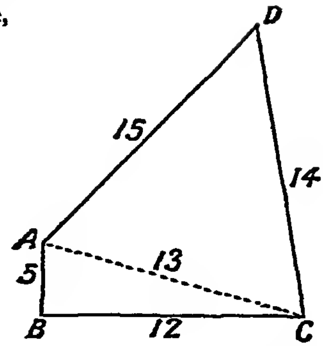
$=\frac{1}{2}(BL+DM) \times AC$

$=\frac{1}{2}$(*sum of perpendiculars*) $\times$ (*diagonal*).

The area of the given quadrilateral $=\frac{1}{2}(2+3) \times 5$

$=12\frac{1}{2}$ sq. ft.

EX. 2. *In the quadrilateral* $ABCD$, $AB=5$, $BC=12$, $CD=14$, $DA=15$, *and the angle* ABC *is a right angle; find the area of the quadrilateral.*



Since the angle B is a right angle, we have

$$AC^2 = 5^2 + 12^2 = 169 = 13^2,$$

therefore $AC = 13$.

The area of the triangle ABC

$$= \tfrac{1}{2} \times 5 \times 12 = 30 \text{ sq. ft.}$$

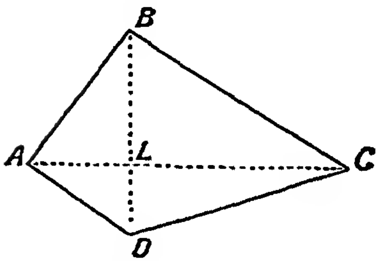
The area of the triangle ACD

$$= \sqrt{21 \times 8 \times 7 \times 6} = 84 \text{ sq. ft.}$$

Hence the area of the quadrilateral

$$= 30 + 84 = 114 \text{ sq. ft.}$$

Ex. 3. *If the diagonals of a quadrilateral be at right angles to each other, its area is equal to half the product of the diagonals.*



Let the diagonals AC, BD intersect at right angles in the point L.

The area of the triangle $ABC \;= \tfrac{1}{2} BL \times AC$,

and the area of the triangle $ADC \;= \tfrac{1}{2} DL \times AC$;

therefore the area of the quadrilateral $= \tfrac{1}{2}(BL + DL) \times AC$

$$= \tfrac{1}{2} BD \times AC.$$

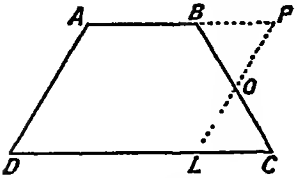
Note.—*Since the diagonals of a rhombus bisect each other at right angles, its area is equal to half the product of the two diagonals.*

104. *The area of a trapezoid is equal to the product of half the sum of the two parallel sides into the perpendicular distance between them.*

Let AB, CD be the two parallel sides. Bisect BC in O, and through O draw POL parallel to AD. Then the parallelogram $APLD$ will be equivalent to the trapezoid $ABCD$.

The two triangles OBP, OCL are equiangular, and therefore similar; and since the sides OB and OC are equal, the remaining sides of the two triangles must also be equal,

each to each. Therefore the triangle OBP is equal to the triangle OCL. Hence the trapezoid $ABCD$ is equal in area to the parallelogram $APLD$.



Since $AP + DL = AB + CD$, therefore $DL = \frac{1}{2}(AB + CD)$. Thus the base of the equivalent parallelogram is equal to half the sum of the parallel sides of the trapezoid, and its height is the same as the height of the trapezoid.

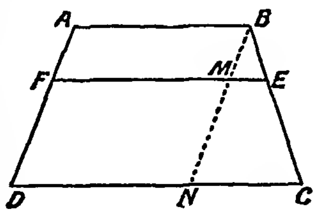
Hence

area of trapezoid $= \frac{1}{2}$(*sum of parallel sides*) $\times$ *height.*

Ex. 1. *The parallel sides of a trapezoid are* 5 *ft. and* 7 *ft., and the perpendicular distance between them is* 4 *ft , find the area.*

The area $= \frac{1}{2}(5 + 7) \times 4 = 24$ sq ft.

Ex. 2. *The parallel sides AB and CD of a trapezoid $ABCD$ are* 5 *ft. and* 8 *ft. respectively. A point E is taken in BC such that CE is double of BE, and EF is drawn across the figure parallel to AB; find the length of EF.*



Draw BMN parallel to AD. Then $DN = AB = 5$ ft., therefore $CN = 3$ ft.

Again, since the triangles BEM, BCN are similar, and $BE = \frac{1}{3}$ of BC, therefore $ME = \frac{1}{3}$ of $CN = 1$ ft. But $EF = FM + ME$, and $FM = AB = 5$ ft.; therefore $EF = 6$ ft.

Note.—The perpendicular from A on CD will be divided by EF in the ratio of 1 : 2. Hence if the length of this perpendicular be given, we can find the areas of the two trapezoids $ABEF$ and $FECD$.

EXERCISES XXXI

In the quadrilateral $ABCD$ the lines BE and DF are drawn at right angles to the diagonal AC; find the area when :

1. $AC=65$ ft., $BE=30$ ft. 9 in., $DF=23$ ft. 3 in.
2. $AC=21$ chs., $BE=16$ chs., $DF=12$ chs.
3. $AC=21$ yds. 2 ft., $BE=9$ yds. 1 ft., $DF=8$ yds. 2 ft.
4. $AC=9$ chs., $BE=4$ chs. 82 lks., $DF=3$ chs. 18 lks.
5. $AC=\frac{1}{2}$ mile, $BE=560$ yds., $DF=231$ yds.
6. $AC=2$ yds. 2 ft. 6 in., $BE=1$ yd. 9 in., $DF=4$ yds. 2 ft. 9 in.
7. $AC=100$ ft., $BE=DF=37$ ft. 6 in.
8. $AC=60$ ft. 10 in., $BE+DF=24$ ft.
9. $AC=21$ yds., $BE=2DF=18$ yds. 2 ft.

[In connection with each of the following questions the student must make an exact diagram, drawn to some convenient scale.]

10. The sides of a quadrilateral are 216, 90, 200, and 188 yds. respectively, and the angle contained by the first two sides is a right angle; find the area.

11. The sides of a quadrilateral are 232, 180, 432, and 320 links respectively, and the angle contained by the second and third sides is a right angle; find the area.

12. In the trapezium $ABCD$ the side $AB=BC=CD=40$ chains, and the side $DA=42$ chains; if the angle at D be a right angle, find the area.

13. $ABCD$ is a quadrilateral; $AB=160$ yds., $BC=78$ yds, $CD=DA=178$ yds; the angle at B is a right angle; find the area.

14. In the kite-shaped figure $ABCD$ the angle at A is of $60°$, $AB=AD=14$ ft., and $BC=DC=25$ ft.; find the area.

15. In the quadrilateral $ABCD$ the angles at B and C are right angles, $AB=BC=240$ yds., and $CD=478$ yds.; find the length of AD and the area of the figure.

16. The diagonals of a quadrilateral are 55 chains and 88 chains, and they intersect at right angles; find the area.

17. The diagonals of a rhombus are 60 and 80 ft. respectively; find the length of a side and the area.

18. The area of a diamond-shaped field is 1 acre; the lesser diagonal is 165 ft.; find the greater diagonal.

19. $ABCD$ is a quadrilateral; $AB=3$ ft., $BC=4$ ft., $CD=DA=6$ ft. 6 in.; if the angle at B is a right angle, find the area.

20. The parallel sides of a trapezoid are 25 yds. 2 ft. 3 in., and 29 yds. 9 in. respectively, and the perpendicular distance between them is 88 yds.; find the area.

21. The parallel sides of a trapezoid are 24 ft. and 37 ft. 8 in., and the area is 370 sq. ft.; find the height.

22. The area of a trapezoid is 10 acres, and the sum of the two parallel sides is 605 yds.; find its altitude.

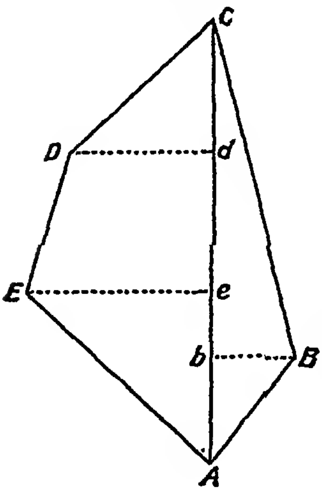
23. A four-sided figure has two sides parallel and two sides equal: one of the equal sides is 6 ft. 6 in., and the longer of the two parallel sides is 15 ft. Find the area, if the distance between the parallel sides is 6 ft.

24. In the last example the two equal sides are produced to meet. Find the area of the smaller of the two isosceles triangles thus formed.

25. The straight line which joins the middle points of the oblique sides of a trapezoid is equal to half the sum of the parallel sides.

Many-sided Figures

105. Suppose the plan of a many-sided field is drawn to scale. We can reduce the plan to a triangle; and then on half the base of the triangle an equivalent rectangle can be constructed, which will have the same height as the triangle (Art. 85). Next we can find the side of a square equivalent to the rectangle (Art. 37). And lastly, the representative fraction of the scale will give us the length of the side of a square field which is equal in area to the given field (Art. 29).



This process for finding the area of a field is seldom used, but it may be employed for checking any gross mistake in the area, as found by the method of the next article.

106. It is usual in practice to measure the parts into which the longest diagonal of a field is divided by the perpendiculars on it from the outlying angular points, and the lengths of these perpendiculars.

We thus obtain a number

of right-angled triangles and trapeziums in which sufficient lengths are known to find their areas.

Ex. *In the field ABCDE the longest diagonal is AC. The following measurements are taken in links:—*

$$Ab=40,\ bB=34;\ Ae=64,\ eE=68;$$
$$Ad=116,\ dD=52;\ \textit{and}\ AC=164:$$

find the area of the field.

Here all the distances along the diagonal are reckoned from A. From the given lengths we have $ed=Ad-Ae=116-64=52$ links, and $dC=AC-Ad=164-116=48$ links.

We have to calculate the areas of two right-angled triangles AeE, CdD, a trapezium $EedD$, and the triangle ABC:

Area of	$AeE = \frac{1}{2}(64 \times 68)$	$= 2176$	square links.
,,	$CdD = \frac{1}{2}(48 \times 52)$	$= 1248$	,, ,,
,,	$EedD = \frac{1}{2}(68+52) \times 52$	$= 3120$	,, ,,
,,	$ABC = \frac{1}{2}(164 \times 34)$	$= 2788$	,, ,,
Therefore the whole area of the field		$= 9332$	,, ,,

Note.—In practical surveying the diagonal is called the **base-line**, or the **chain-line**; and the perpendiculars which rise from it to the right and left are called respectively the right and left **off-sets**. All the measurements are recorded in a **Field Book**, each page of which is divided into three columns. Beginning at the bottom of a page, the distances from the starting station measured along the base-line are put down in the middle column; and the lengths of the offsets are written in the right and left columns against the distances on the base-line at which they rise.

Thus the measurements in this example will be recorded as follows:—

	Links.	
	To C	
	164	
D 52	116	
E 68	64	
	40	34 B
	From A	

107. Sometimes it is necessary to have more than one base-line. Thus in the case of a field which is very nearly

triangular in shape it will be found convenient to measure three base-lines.

Ex. *Draw a plan of a field to the scale of 3 chains to the inch from the following notes, and find the area, the lengths being given in links.*

	To ⊙ *A*	
	644	
	450	50 *G*
	From ⊙ *F*	
	To ⊙ *F*	
	666	
E 100	425	
D 80	150	
	From ⊙ *C*	
	To ⊙ *C*	
	725	
	250	100 *B*
Begin	at ⊙ *A*	range East.

The starting stations are marked with a small circle in the field-book.

We first find the area of the triangle formed by the base-lines: and then add the areas which lie outside this triangle, and subtract those which fall inside.

By Art. 86 the area of the triangle $ACF = 197,670$ square links nearly.

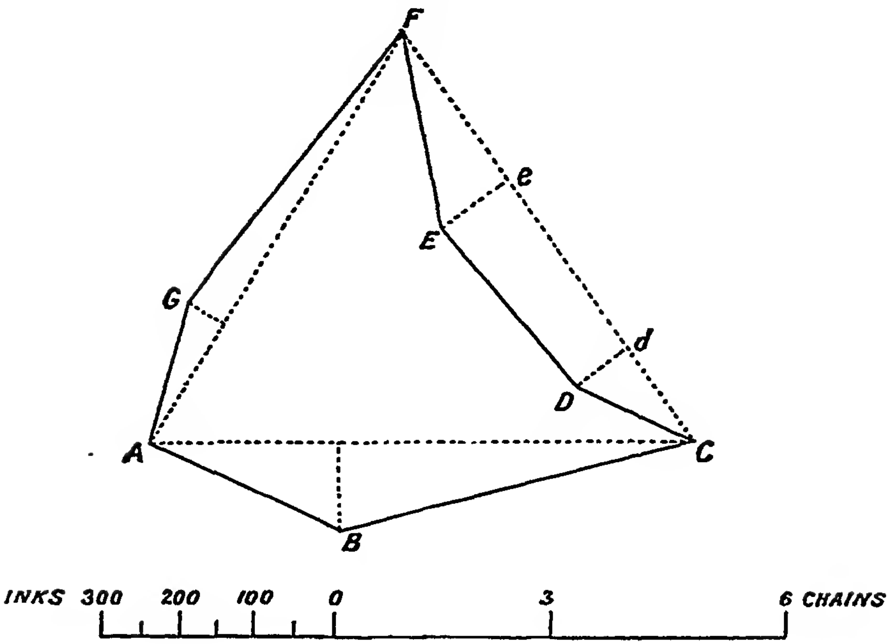
The area of the triangle $ABC = \frac{1}{2}(100 \times 725) = 36,250$ square links.
" " " $AGF = \frac{1}{2}(50 \times 644) = 16,100$ " "
" " " $CDd = \frac{1}{2}(80 \times 150) = 6,000$ " "
" " " $EeF = \frac{1}{2}(100 \times 241) = 12,050$ " "
" " trapezoid $DdeE = \frac{1}{2}(180 \times 275) = 24,750$ " "

Hence the area of the field

$$= \left.\begin{matrix} 197,670 \\ +36,250 \\ +16,100 \end{matrix}\right\} - \left\{\begin{matrix} 6,000. \\ +12,050. \\ +24,750. \end{matrix}\right.$$

$$= 250,020 - 42,800 = 207,220 \text{ square links.}$$

$$= 2 \text{ acres, } 12 \text{ poles nearly.}$$



EXERCISES XXXII

Draw to scale the plans of the following fields, and find their areas in acres, roods, and poles:—

1.

	Yards.	
	To D	
	1300	
C 300	900	
	400	300 B
	From A	

2

	Chains.	
	To C	
	20	
D 8.66	15	
	5	8.66 B
	From A	

3.

	Chains	
	To *D*	
	18.72	
	11.5	7.2 *C*
E 6.2	9	
	3	4.5 *B*
	From *A*	

4.

	Yards	
	To *D*	
	1504	
C 315	1200	
	800	386 *E*
B 425	720	
	From *A*	

5.

	Links.	
	To *D*	
	1440	
E 280	1085	
	880	588 *C*
F 420	588	
	295	412 *B*
	From *A*	

6.

	Links.	
	To *D*	
	1040	
	600	320 *C*
E 900	560	
	400	780 *B*
F 500	200	
	From *A*	

7.

	Links.	
	To *E*	
	800	
	650	375 *D*
	400	720 *C*
	150	375 *B*
	From *A*	

8.

	Links.	
	To *D*	
	750	
	600	325 *C*
	340	210 *E*
	100	250 *B*
	From *A*	

9.

	Links.	
	To $\odot$ *A*	
	1300	
F 80	880	
	From $\odot$ *E*	
	To $\odot$ *E*	
	1300	
	465	50 *D*
Turn	at $\odot$ *C*	to the left.
	To $\odot$ *C*	
	1000	
	300	100 *B*
Begin	at $\odot$ *A*	range East.

10.

	Links.	
	From *G*	go to *A*
	To $\odot$ *G*	
	300	
	150	50 *F*
	80	150 *E*
	From $\odot$ *D*	go North.
	To $\odot$ *D*	
	400	
	280	75 *C*
	120	50 *B*
Begin	at $\odot$ *A*	go East.

11. Each side of a square board is divided into 5 equal parts. On each middle division a square is described and sawn off. Make a correct figure of the remaining part, and compare its area with that of the original board.

12. The corners of an equilateral triangle are cut off, so as to leave in each case one-third of the side of the original triangle. Compare the area of the six-sided figure thus obtained with the area of the equilateral triangle.

13. In the five-sided field *ABCDE* the sides *ED*, *DC*, *CB* are 1000, 2400, and 1700 yds. respectively; the sides *AB* and *AE* are equal; the angles at *C* and *D* are right angles, and the angle at *A* is of 60°. What price must be paid for the field at Rs.100 per acre?

14. One corner of a rectangle is cut off by the straight line which joins the middle points of two adjacent sides. Show that the area of the remaining figure is to that of the original as 7 : 8.

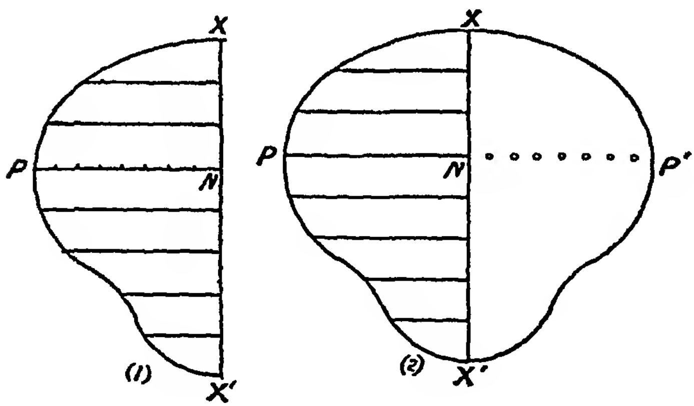
15. On each side of an equilateral triangle, and lying inwards, an isosceles triangle is constructed whose height is one-sixth of the height of the equilateral triangle. Show that the area of the star-shaped figure thus formed is half that of the original triangle.

CHAPTER XI

SYMMETRY AND LOCI

Symmetry

108. **Fold** a piece of paper and cut it of any shape as in Fig. 1. Rule a series of parallel lines on the folded paper, all perpendicular to the crease XX'.



Take one of the parallel lines, such as PN, and prick a series of holes in it which pass through both folds of the paper.

Now unfold as in Fig. 2, and notice that this figure is

capable of being folded about the line XX' so that one part exactly fits on the other.

Note also that for every point, such as P, on one side of the line XX' there is another point P' at an equal distance on the other side, so that XX' is the perpendicular bisector of PP'.

A figure which can be folded about a line so that one part falls exactly on the other is said to be **symmetrical** *about the line, and the line is called the* **axis of symmetry** *of the figure.*

The points P, P' are called **corresponding points.**

The student ought to bear in mind the following two facts about a symmetrical figure:—

When a figure can be folded about an axis so that one half exactly fits the other, then all lines drawn across the figure at right angles to the axis are bisected by the axis.

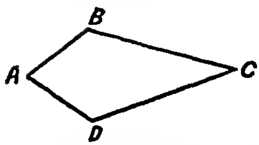
Conversely, when the boundary of a figure consists of pairs of points, the joins of which are bisected at right angles by some line in the figure, then the figure is capable of being folded about this line so that its two parts fall on each other exactly.

EXERCISES

Prove the following cases of symmetry by drawing, cutting, and folding, and also by measuring the two halves of perpendiculars to the axis of symmetry drawn across the figure :—

1. A circle about any diameter.
2. An isosceles triangle about a perpendicular from the vertex on the base.
3. A rectangle about two lines through its centre parallel to the sides.
4. A square about both diagonals, and about lines through its centre, as in 3.
5. A rhombus about both diagonals.

6. A kite-shaped figure ($AB=AD$, $CB=CD$) about its longest diagonal.



7. An equilateral triangle about three different lines.

8. A semicircle about a certain radius.

9. Two unequal intersecting circles about the line through their centres.

10. Two equal circles about two lines.

Loci

109. When a point moves subject to some given geometrical condition, the path which it describes is called the **locus of the point.**

The student has already come across some loci. Thus:—

(i.) *If a point move so as to remain at the same distance from a fixed point in the plane of its motion, the locus of the point is a circle* (Art. 7).

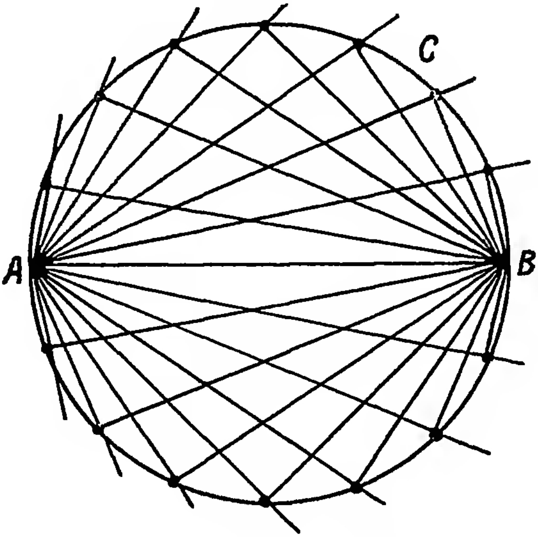
(ii.) *If a point move so as to remain at the same distance from a fixed straight line the locus of the point is two straight lines parallel to the fixed line and on either side of it* (Art. 14).

(iii.) *The locus of the vertices of all triangles having a given area, and described on the same side of a given base, is a straight line parallel to the base* (Art. 85).

Again, let us find the locus of the vertices of all right-angled triangles which have a given hypotenuse AB.

Through A draw a number of straight lines in different directions, and through B draw perpendiculars to each of these lines. We have to find the locus of the feet of the perpendiculars. It will be found that a circle described on AB as diameter passes through the feet of all the perpendiculars. Hence we see that *the locus of the vertices*

of all right-angled triangles having a given hypotenuse is the circle described on the hypotenuse as diameter.



110. Conversely, if we take a point C on the circumference of a circle and join it to the extremities of a diameter AB, the angle ACB will be found on measurement to be a right angle. An angle such as ACB which a diameter subtends at a point on the circumference of a circle is called the **angle in a semicircle.** Hence it follows that *the angle in a semicircle is a right angle.*

We shall give two more examples of loci, which the student will find of great use in his future work.

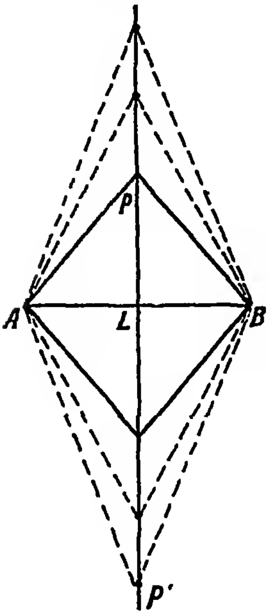
111. *The locus of a point which is equidistant from two fixed points is the perpendicular bisector of the straight line joining the two fixed points.*

Let A, B be two fixed points.

Bisect AB in L, and draw PLP' perpendicular to AB. Then every point on PLP' is equidistant from the points A and B.

For, take any point P on the perpendicular bisector and join PA, PB.

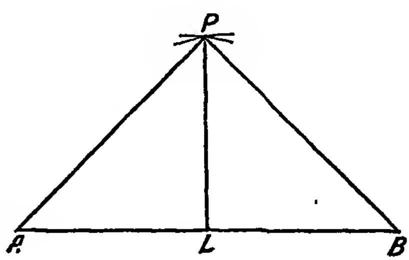
Fold the figure about the line PLP'. Then, since the angles at L are equal, and the line LB is equal to LA, therefore B will fall on A.



Hence PA and PB will coincide. Similarly any other point on the perpendicular bisector is equidistant from A and B.

Hence every point on the perpendicular bisector of AB is equidistant from A and B. We can further show that any point which is equidistant from A and B must lie on the perpendicular bisector of AB.

With A and B as centres, and any the same radius, describe arcs intersecting in P. Then PA, PB are equidistant from A and B. Bisect AB in L, and join LP.



Then PL will be perpendicular to AB, for the triangles PAL, PBL have three sides of the one equal to three sides of the other; hence they are congruent (Art. 56).

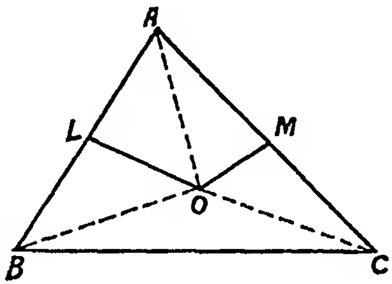
We thus see that every point on the perpendicular bisector of AB is equidistant from A and B, and that

every point equidistant from A and B lies on the perpendicular bisector of AB.

Therefore *the locus of a point equidistant from the extremities of a given line is the perpendicular bisector of the line.*

Ex. 1. *To find a point equidistant from three given points A, B, and C.*

Draw LO the perpendicular bisector of AB and MO the perpendicular bisector of AC. Let these lines intersect in O. Then O is equidistant from A, B, and C.



Since O lies on the perpendicular bisector of AB, we have $OA = OB$.

And since O lies on the perpendicular bisector of AC, we have $OA = OC$.

Hence $OA = OB = OC$.

Ex. 2. Draw a triangle whose sides are 13, 14, and 15 centimetres, and describe a circle which will pass through its angular points.

Ex. 3. Construct a triangle whose sides are 3″, 4″, and 5″; and describe a circle which will pass through its angular points.

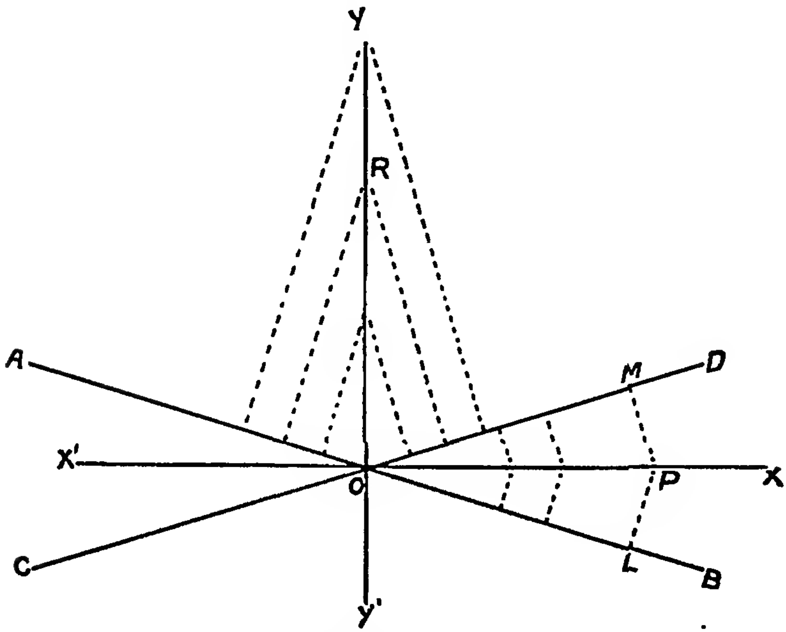
Where is the centre of this circle? Why may you expect to find it in this position?

112. *The locus of a point which is equidistant from two intersecting straight lines consists of the pair of straight lines which bisect the angles between the two given lines.*

Draw the lines AOB, COD, and also draw the lines XOX', YOY' which bisect the acute and obtuse angles respectively between these lines.

Then the perpendicular distances from OB and OD of every point on the line XOX' are equal. For, take any point P on OX and draw PL, PM perpendiculars on OB and OD.

Then since the angles *POM*, *POL* are equal, so are their complements *OPM* and *OPL*. Now if the figure



be folded about the line *XOX'*, then *OB* will fall on *OD* and *PL* on *PM*, since the angles at *O* and *P* are equal. Hence *PL* and *PM* will coincide. The same can be proved of any other point on the line *OX*.

Therefore every point on *OX* is equidistant from *CB* and *OD*.

Similarly by folding the figure about the line *YOY'* it can be proved that every point on the line *YOY'* is equidistant from the lines *AOB*, *COD*.

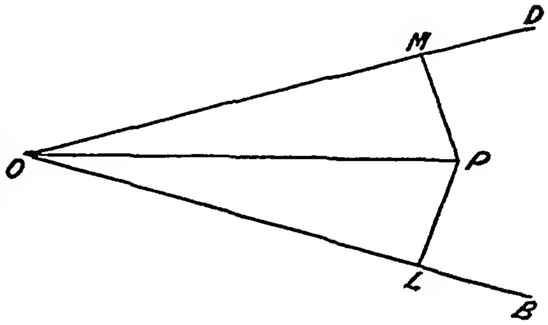
Next cut off $OL = OM$, and draw *LP*, *MP* at right angles to *OB*, *OD* meeting in *P*.

Then LP will be found equal to MP.

Now join OP. Then OP is the bisector of the angle BOD.

For in the two triangles POL, POM the sides of the one are equal to the sides of the other.

Hence the triangles are congruent, and the angle POM is equal to the angle POL.



We have thus proved that if the perpendiculars from any point P on the lines OB, OD are equal, the point lies on the bisector of the angle between OB and OD.

Thus the proposition of this article is completely established.

Ex. 1. Find a point equidistant from three given lines forming a triangle.

Ex. 2. Construct a triangle whose sides are 2.6″, 2.8,″ and 3″. Find a point equidistant from the three sides, and measure this distance.

Ex. 3. Construct a triangle whose sides are 7 cm., 8 cm., and 9 cm.; find a point equidistant from the three sides, and with this point as centre, and its perpendicular distance from a side as radius, describe a circle. Notice that this circle lies wholly inside the triangle, and meets each side in one point only.

Ex. 4. Repeat the last exercise with a triangle whose sides are $3\frac{1}{2}''$, $4''$, and $4\frac{1}{4}''$, also with an equilateral triangle of side 5″.

Ex. 5. Draw AB four inches in length and construct the locus of the vertices of triangles described on one side of AB and having an area of 12 sq. in.

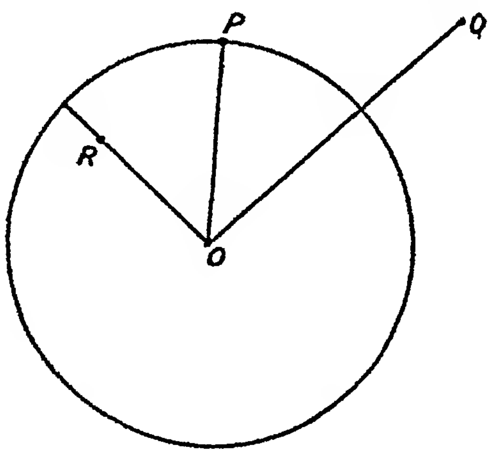
Ex. 6. Describe an equilateral triangle of 3″ side, and find its orthocentre. Explain why this point is equally distant from the three sides of the triangle.

CHAPTER XII

CIRCLE

113. We have already defined a circle and made use of its principal property, namely, that all lines drawn from the centre to the circumference are equal to one another. We shall now consider the properties of this figure in detail.

114. Draw a circle and take three points, one lying



without it, another on the circumference, and a third within it.

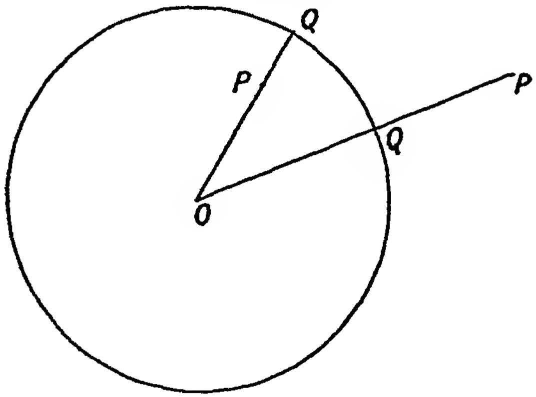
If O be the centre of the circle it is clear from the figure that—

OQ is greater than the radius,
OP is equal to the radius,
and OR is less than the radius.

Hence *a point lies without, upon, or within the circumference of a circle according as its distance from the centre is greater than, equal to, or less than the radius.*

115. *Two circles whose radii are equal are congruent.* If this is not evident, describe two circles having the same radius, cut them out, and place one on the top of the other so that their centres coincide. The two circles will fit together exactly.

116. *The distance of a point from a circle is its distance from the circumference measured in the direction of the radius.*



Thus if P be a point in the plane of a circle whose centre is O, the distance of P from the circle is PQ, where Q is the point in which OP meets the circumference.

117. *Circles which have the same centre are called* **concentric circles.**

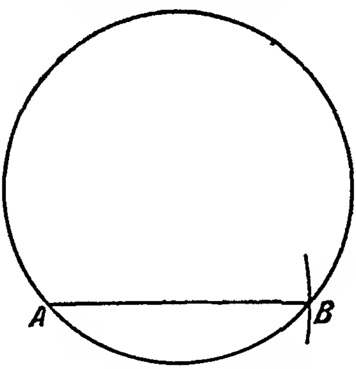
Ex. 1. Describe three concentric circles of radii $\frac{3}{4}''$, $1''$, and $1\frac{1}{2}''$ respectively. Show that every point on any one of the circles is at a constant distance from the other two.

Ex. 2. Describe three concentric circles of radii $1''$, $1\frac{1}{2}''$, and $2''$ respectively. Show that every point on the middle circle is equidistant from the other two.

Ex. 3. The locus of a point which moves so as to remain at a constant distance from the circumference of a fixed circle is two concentric circles.

118. *The* **chord** *of a circle is the straight line joining two points on its circumference.*

Being given a circle, we can always place in it a chord of any given length less than the diameter.



Open out the compasses till the points are the given length apart.

Place the needle point at A and describe an arc cutting the circle in B.

Then AB is the required chord.

Notice that when the given length is not less than the diameter the arc will not cut the circle.

Ex. 1. Describe a circle with any radius and place in it six consecutive chords, each equal to the radius.

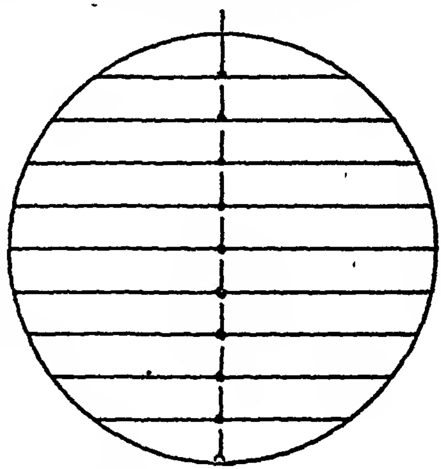
Ex. 2. Describe a circle whose radius is $2.5''$ and place in it two consecutive chords of $3''$ and $4''$. Verify that the line joining the extremities of the chords passes through the centre.

Ex. 3. Describe a circle of $2''$ radius and place in it a chord of $2''$. Join the middle point of the chord to the centre, and verify that the joining line is perpendicular to the chord.

Ex. 4. Describe a circle with any radius and draw in it two parallel chords. Verify that the line joining the middle points of the chords passes through the centre.

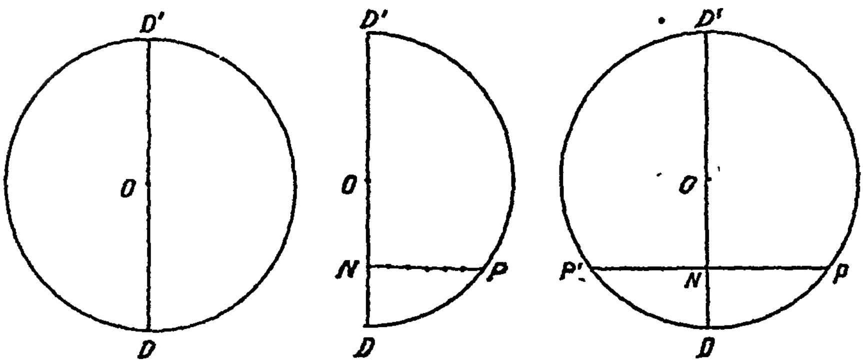
Ex. 5. Draw a circle of 2″ radius and rule a series of parallel chords. *Find the middle points of these chords, and show that they all lie on a diameter perpendicular to the chords.*

Ex. 6. Take a circle of $1\frac{1}{2}''$ radius, and place in it a chord of 1″. Draw the diameter through the middle point of this chord, and rule a number of chords parallel to the first. Verify that each of the chords is bisected by the diameter.



119. *A circle is symmetrical with respect to any diameter.*

Describe a circle with any convenient radius, and draw a diameter. Cut out the figure, and fold it about the diameter. Then one semicircle will fall exactly on the other. This is necessary, for all lines drawn from the centre to the circumference are equal.



Take any point P on the circumference of the folded figure and draw PN perpendicular to the diameter DD'.

Prick a series of holes with a pin all along the line PN. Now unfold, as in the third figure, and notice that—

(i.) $PN = P'N$.

(ii.) The angles at N are right angles, and consequently (iii.) the diameter DD' is the perpendicular bisector of the chord PP'.

From this experiment we conclude that—

The perpendicular bisector of a chord passes through the centre.

A circle is symmetrical with respect to a diameter.

For, corresponding to every point P on one side of the diameter DD', there is another point P' at an equal distance on the opposite side.

Ex. 1. *Being given a circle, find its centre.*

Draw any chord and its perpendicular bisector; the middle point of the latter is the centre.

Ex. 2. *Given the arc of a circle, find its centre.*

Draw any two chords and their perpendicular bisectors. The point of intersection of the latter is the centre.

120. *A straight line drawn from the centre of a circle to bisect a chord is at right angles to the chord; conversely, the perpendicular to a chord from the centre bisects the chord.*

A little consideration will show that these results also follow from the experiment of the last article. We can also argue them out independently.

Let PP' be a chord of a circle whose centre is O.

Find N the middle point of PP' and join ON.

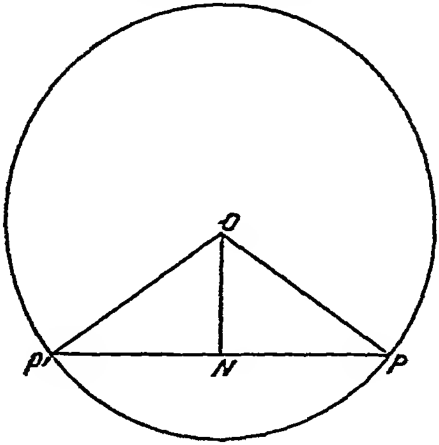
Then ON will be perpendicular to PP'.

For in the triangles ONP, ONP' we have

$$\left.\begin{aligned} OP &= OP', \\ PN &= P'N, \\ ON &= ON. \end{aligned}\right\}$$

Hence the triangles are congruent, and the angles at N are equal.

Therefore ON is at right angles to PP'.



Again, let ON be drawn at right angles to PP'. Then ON will bisect PP'.

For in the triangles ONP, ONP',

$$\left.\begin{array}{c} OP = OP', \\ \text{the angle at } P = \text{the angle at } P', \\ \text{and the angles at } N \text{ are equal.} \end{array}\right\}$$

Hence the triangles are congruent, and $NP = NP'$.

Therefore ON bisects the chord PP'.

EXERCISES XXXIII

1. The radius of a circle is 5 cm.; place in it a chord of 8 cm., and find the distance of the middle point of this chord from the centre.

2. In a circle of 2.5″ radius place a chord of 1.4″, and find, both by measurement and calculation, its distance from the centre of the circle.

3. Describe a circle of radius 13 cm., and from the centre draw a straight line 5 cm. in length. At the other extremity of this line erect a perpendicular chord, and find its length.

4. Draw a circle of 2″ radius. Through the point of bisection of any radius draw a perpendicular chord, and find its length correct to two places of decimals.

5. Describe two circles of one inch radius, and place in each a chord one inch long. Join the extremities of the chords to the centres, and measure the angles which each chord subtends at the centre.

6. Describe two circles of 2″ radius, and in each draw a pair of radii containing an angle of 120°. Find the lengths of the chords which join the extremities of the radii. Are they equal? Are their distances from the centres equal?

7. In a circle of 3″ radius place two chords, each measuring 2″, and measure the angles which they subtend at the centre. Are they equal?

8. Take a circle of 5 cm. radius and draw two pairs of radii, each pair containing an angle of 45°. Measure the chords which join the extremities of each pair of radii.

9. Describe two circles of 7 cm. radius, and in each draw a pair of radii containing an angle of 75°. Cut out the figures bounded by the radii and the arc included between them. Fit them together, and see that the arcs are equal.

10. In a circle of 1.5″ radius place two consecutive chords, each measuring 1.5″. Join the extremities of the chord by a straight line. Place three chords equal to this line in the circle. Measure the angles which the last three chords subtend at the centre of the circle.

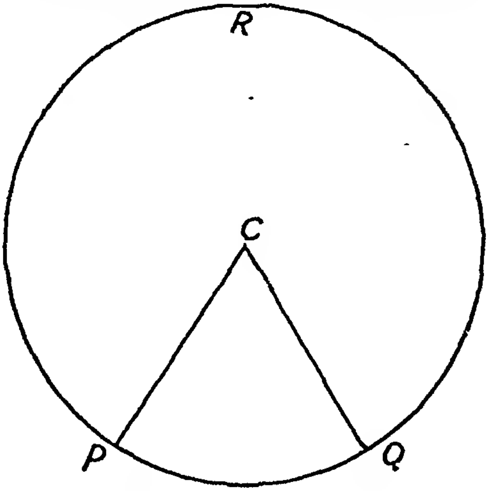
121. *In equal circles (or, in the same circle)* (i.) *if two arcs subtend equal angles at the centres, they are equal;* (ii.) *conversely, if two arcs are equal, they subtend equal angles at the centres.*

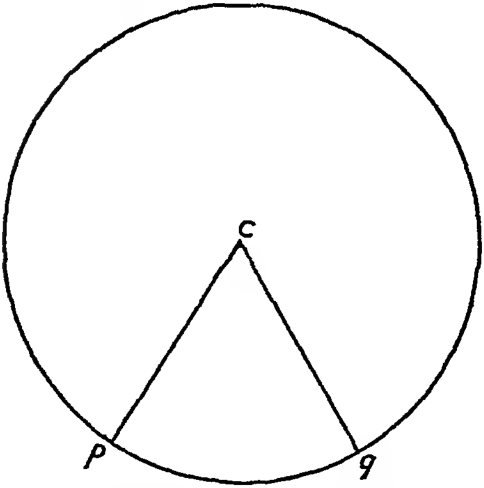
(i.) Describe two equal circles, and in each draw a pair of radii containing the same angle (say of 60°).

Then the arcs PQ, pq subtending equal angles at the centre will be equal.

Cut out the circle PQR and place it on the other so that the centres coincide, then the circumferences will also coincide. Now make CP coincide with cp; then since the angle $PCQ = pcq$, CQ will coincide with cq, and hence the arc PQ will coincide with pq.

Hence, *in equal circles, if the angles at the centre are equal the arcs on which they stand are also equal.*





(ii.) Next suppose the arcs PQ, pq are equal, the angles at C and c will also be equal.

For, placing the circles as before, their circumferences will coincide.

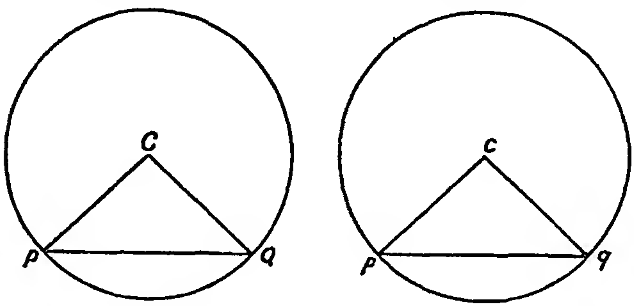
Now if P be made to coincide with p, then since $PQ = pq$, the point Q will coincide with q.

Hence the lines CP, CQ will coincide with cp, cq; *i.e.* the angle PCQ will coincide with the angle pcq.

Therefore *in equal circles equal arcs subtend equal angles at the centre.*

A little consideration will show that these results hold good for arcs and angles of the same circle.

122. *In equal circles* (*or, in the same circle*) (i.) *if two chords are equal, they cut off equal arcs;* (ii.) *conversely, if two arcs are equal, the chords of the arcs are equal.*



(i.) In the triangles CPQ, cpq,

$$\left.\begin{aligned} CP &= cp, \\ CQ &= cq, \\ PQ &= pq. \end{aligned}\right\}$$

Hence the triangles are congruent (Art. 56). Therefore the angles at C and c are equal; whence by Art. 121 it follows that the arcs PQ, pq are equal.

(ii.) If the arc PQ equals the arc pq, then by Art. 121

$$\left.\begin{aligned} &\text{the angle } C = \text{the angle } c; \\ \text{also} \quad & CP = cp, \\ & CQ = cq. \end{aligned}\right\}$$

Therefore by Art. 56 the triangles PCQ, pcq are congruent. Hence the chord PQ = the chord pq.

The **sector** *of a circle is the figure contained by two radii and the arc included between them.*

A **segment** *of a circle is the figure bounded by an arc and the chord of the arc.*

EXERCISES

1. From a circle of 2″ radius cut off two sectors whose angles are each 45°. Fit them together, and see that they are equal.

2. From a circle of 1.5″ radius remove two sectors of 120°; fit them together, and verify that they are equal.

3. Describe two circles of 5 cm. radius; in each place a chord of 7 cm. Cut along the chords, and show that the smaller segments are equal, as are also the larger.

4. From a circle of 1″ radius cut off six sectors, each containing an angle of 60°. Show that all the sectors are equal.

5. Describe a circle of 6 cm. radius, and place in it two chords of 7 cm. each. Measure the distances of their middle points from the centre of the circle. Are the distances equal?

6. In a circle of 2½″ radius place two chords of 3″. Find the distances of the chords from the centre.

7. Describe a circle of 2″ radius. From the centre draw three straight lines at random, each ¾″ long. Through the extremities of these lines draw perpendicular chords. Measure them, and verify that they are equal.

8. Take a circle of 2″ radius. From the centre draw three lines, each 1″ long, making angles of 120° with one another. Through the extremities of these lines draw perpendicular chords, and show that they are equal.

9. In a circle of 3″ radius place chords of 1″, 2″, 3″, 4″, and 5″. Measure the distances of the chords from the centre, and verify that the greater the chord is the shorter is its distance and *vice versa*.

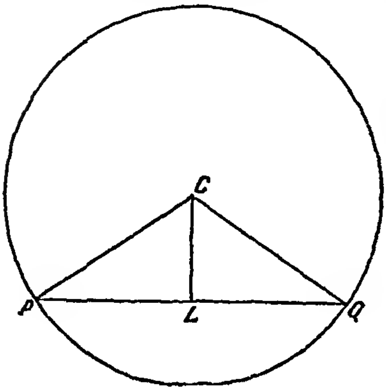
10. In a circle of 7 cm. radius draw a series of parallel chords, and draw also the diameter which bisects them all at right angles. Verify that the chords nearer the centre are greater than those more remote.

123. From the exercises given above the student must have seen that—

Equal chords of a circle are equidistant from the centre; conversely, chords which are equidistant from the centre are equal.

We can view this proposition in another light.

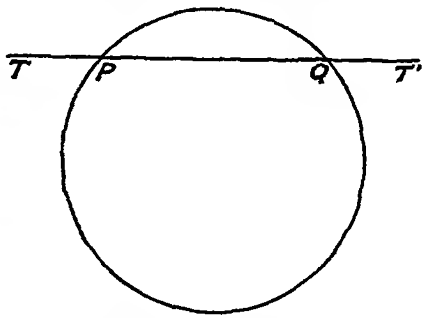
Let L be the middle point of the chord PQ. Then CL is at right angles to PQ.



Hence CPL is a right-angled triangle. Thus the radius, the half-chord, and the distance of the chord from the centre form a right-angled triangle.

Now if two sides of a right-angled triangle are given the third side is known.

Hence it follows that for the same lengths of radius and half-chord we must have the same perpendicular distance from the centre; and conversely, for the same lengths of radius and perpendicular distance we must have the same lengths of half-chords or chords.



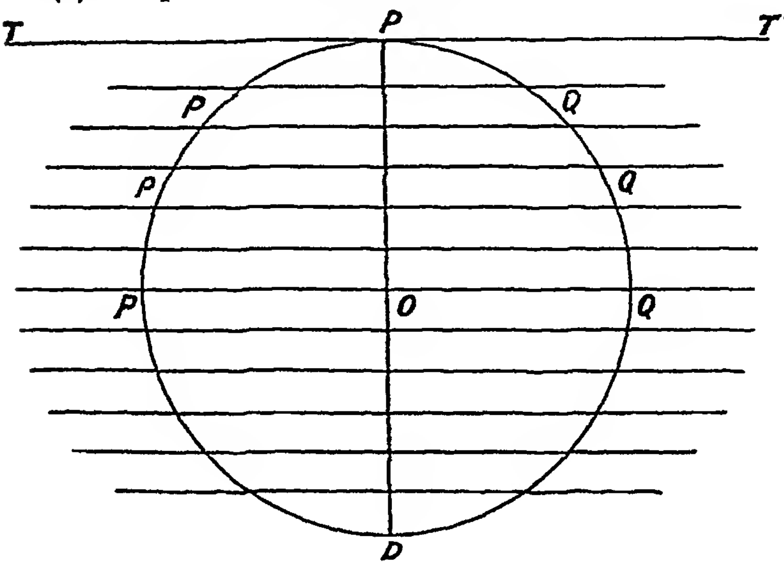
124. *A straight line which cuts a circle is called a* secant.

Thus $TPQT'$ is a secant, cutting the circle in P and Q.

Stretch a fine thread across a circle, and move it parallel to itself, as in the

figure, or simply draw a series of parallel secants. Now notice the following points in this operation :—

(1) The portions of the thread, such as *PQ*, intercepted



by the circle are all bisected by the diameter *PD*, which is perpendicular to them. (Art. 119.)

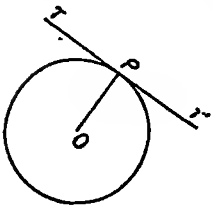
(2) The longest intercept is that which passes through the centre, and as you proceed farther from the centre the intercepts become shorter and shorter, and the points *P* and *Q* approach nearer to one another, so that when the thread is passing the extremity of the diameter these points become coincident and the thread appears to touch the circle.

In this last position the secant *TT'* is called a **tangent** (or touching line) to the circle at the point *P*.

Since the line *OP* remains perpendicular to the secant throughout, it must be perpendicular to it in its ultimate position *TT'*. Hence—

The tangent at any point of a circle and the radius through the point are perpendicular to one another.

125. From the above we derive the following method for drawing a tangent to a circle at any point P.



Draw the radius OP.

Through P draw TPT' at right angles to OP. Then TT' is the required tangent.

EXERCISES

1. Describe a circle of 6 cm. radius, and draw 3 radii making angles of 120° with one another. Draw tangents to the circle at the extremities of these radii. Measure the sides and angles of the triangle formed by the tangents, and thus determine its species.

2. Describe a circle of 2″ radius, and in it draw two diameters at right angles. Draw tangents to the circle at the extremities of these diameters, and discover the nature of the four-sided figure formed by the tangents.

3. Take a circle of 1.5″ radius and draw radii at angular intervals of 72°. Draw the tangents at the extremities of these radii. The five-sided figure thus formed has all its sides equal and also all its angles equal.

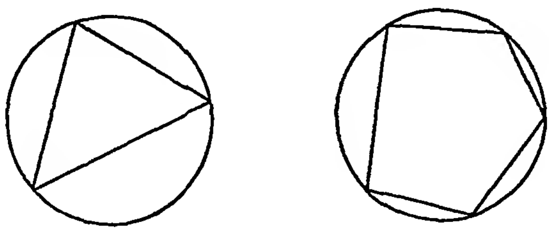
4. In a circle of 4 cm. radius draw six radii at angular intervals of 60°. Draw the tangents at the extremities of the radii, and examine the six-sided figure thus formed. Are its sides and angles equal?

5. In a circle of 1.6″ radius draw two diameters at right angles, and then draw the four radii which bisect the angles between them. Draw tangents at the extremities of all the radii in the figure, and show by measurement that all the sides and angles of the eight-sided figure formed by the tangents are equal.

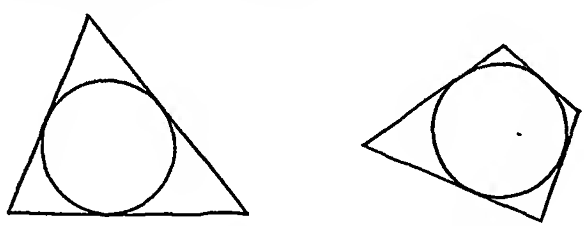
6. In the figures you have constructed for the last five exercises join each point of contact of a tangent with the adjacent points of contact, you will thus obtain five other figures such that all their sides and angles are equal.

Inscribed and Circumscribed Figures

126. *If the angular points of a rectilineal figure lie on the circumference of a circle, the figure is said to be* inscribed *in the circle, and the circle is said to be* described about *the figure.*



If each side of a rectilineal figure touches a circle, the circle is said to be inscribed *in the figure, and the figure is said to be* described about *the circle.*



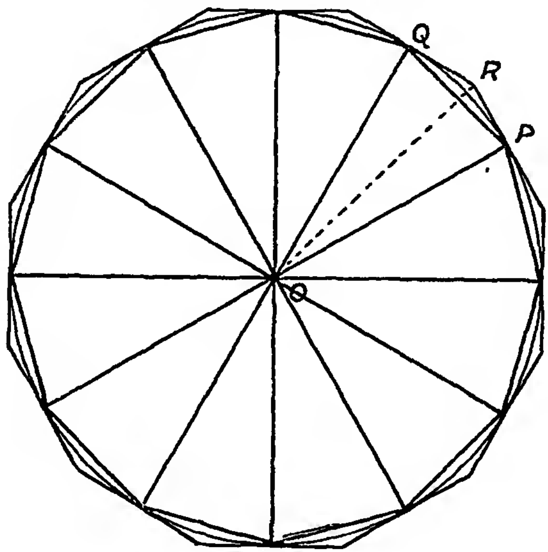
DEFINITIONS.—*A rectilineal figure contained by more than four lines is called a* polygon.

A polygon of five sides is called a pentagon ; *of six sides, a* hexagon ; *of seven sides, a* heptagon ; *of eight sides, an* octagon ; *of ten sides, a* decagon ; *and one of twelve sides, a* dodecagon.

A polygon which is both equilateral and equiangular is said to be regular.

Thus a regular hexagon is one in which the six sides are equal, and the six angles are also equal.

The exercises of the last article supply us with a method for inscribing or circumscribing a regular polygon of any number of sides in or about a given circle. Thus to inscribe a regular dodecagon in a circle we divide the angular space about the centre into twelve equal parts.



In other words, we draw radii at angular intervals of $360° \div 12$, *i.e.* of $30°$.

Then by joining the ends of these radii we obtain a twelve-sided figure inscribed in the circle.

Since the angles at the centre are all equal, the chords on which they stand are all equal.

Hence the figure has all its sides equal.

Again, considering an isosceles triangle formed by two radii and a chord, we see that its vertical angle is of $30°$. Hence each angle at the base is of $75°$.

But an angle of the dodecagon is made up of two such angles.

Therefore each angle of the dodecagon is of 150°.

Thus the figure has all its angles equal.

Hence the figure constructed is a regular dodecagon.

In practice it is not necessary to draw all the twelve radii. It is sufficient to draw two radii OP, OQ containing an angle of 30°. Then the chord PQ can be stepped all round the circumference.

To describe a regular dodecagon about a given circle, draw tangents to the circle at the angular points of the inscribed dodecagon.

EXERCISES

1. Draw a circle of 2″ radius, and place successive chords equal to the radius all round the circumference. This will give an inscribed regular hexagon.

2. In the last exercise join the alternate angular points of the hexagon; this will give the inscribed equilateral triangle.

3. Construct a regular hexagon in and about a circle of 2.4″ radius.

4. Construct an equilateral triangle in and about a circle of 4 cm. radius.

5. Inscribe an equilateral triangle in a circle of 2″ radius; and on the sides of the triangle, and lying externally to it, describe semicircles.

6. Circumscribe an equilateral triangle to a circle of 1.8″ radius.

7. In a circle of 2″ radius inscribe a regular hexagon, and taking the angular points of the hexagon as centres describe circles of 1″ radius.

8. Take a circle of 1.8″ radius, and draw two perpendicular diameters; the joins of the extremities of these diameters will form the inscribed square.

9. Construct squares in and about a circle of 2.5″ radius.

10. Inscribe a square in a circle of 5 cm. radius, and on the sides of the square describe semicircles lying outside the square.

11. In a circle of 1.6″ radius inscribe a square; with the angular points of the square as centres, and with radii equal to half the side of the square, describe quadrants lying inside the square.

(*N.B.*—A quadrant is the fourth part of a circle.)

12. About a circle of 2″ radius describe a square, and draw quadrants as in the last exercise.

13. Take a circle of 2″ radius and draw two radii containing an angle of 72°. The join of the extremities of the radii is the side of a regular pentagon inscribed in the circle.

14. Construct a regular pentagon in and about a circle of 1.8″ radius.

15. Inscribe a pentagon in a circle of 5 cm. radius; join the angular points by straight lines in every possible way. These lines will form another regular pentagon.

16. In a circle of 1.3″ radius draw two radii at right angles, and draw the bisectors of the angles between the radii; you have now obtained the angular points of the inscribed regular octagon.

17. Construct a regular octagon in and about a circle of 1.5″ radius.

18. In a circle of 2″ radius inscribe a regular octagon; with the angular points as centres and with radii equal to half a side of the octagon describe portions of circles lying inside the octagon.

19. Construct a regular dodecagon in and about a circle of 6 cm. radius.

20. Inscribe a regular dodecagon in a circle of 2″ radius; in the same circle inscribe a regular hexagon and an equilateral triangle.

21. Take a circle of 4.5 cm. radius; step round the circumference with the compasses dividing it, by repeated trial, into five equal parts.

This is one of the best methods for constructing an inscribed regular figure of any number of sides.

22. In a circle of 3 cm. radius inscribe a regular heptagon, by stepping out with the compasses.

23. In a circle of 1.4″ radius inscribe a regular polygon of nine sides.

24. In a circle of 1″ radius inscribe a regular pentagon.

25. Take AB 1.7″ long, and on it describe an equilateral triangle AOB; with O as centre and radius OA describe a circle; place the chord AB all round the circumference. You have now constructed a hexagon of which a side is given.

26. On a base 3.5 cm. long construct a regular hexagon.

27. On a base 1″ long construct a regular hexagon, and on the sides of this hexagon construct six other regular hexagons.

28. Take AB 2″ long; produce AB to C and cut off $BC=1''$; draw CL perpendicular to BC and 1″ long; join BL and from it produced cut off $BD=BA$; draw the perpendicular bisectors of AB and BD to meet in O; with centre O and radius OA describe a circle, and place AB all round the circumference.

We have now constructed a regular octagon on a given base AB.

Join PQ. Then PQ is called the **chord of contact** of tangents drawn from T.

By folding the figure about CT it is seen to be symmetrical about this line.

Hence it follows that—

(i.) $TP = TQ$, i.e. *the two tangents drawn from an external point are equal.*

(ii.) The angle PTC = the angle QTC, i.e. *the two tangents are equally inclined to the line TC.*

(iii.) $PL = QL$, and the angle PLT = the angle QLT, i.e. *CT bisects the chord of contact at right angles.*

The student ought particularly to notice that the triangle CPT is a right-angled triangle, and PL is the perpendicular from the right angle on the hypotenuse. Hence the theorem of Art. 97 is applicable here.

Ex. *If $CP = 5''$, and $CT = 13''$; find the length of each tangent and of the chord of contact.*

From the right-angled triangle CPT

$$PT^2 = CT^2 - CP^2$$
$$= 169 - 25 = 144;$$
$$\therefore PT = 12''.$$

Again, since PL is the perpendicular from the right angle on the hypotenuse, the triangles CPL, CPT are similar.

$$\therefore \quad \frac{PL}{CP} = \frac{PT}{CT},$$

i.e. $$\frac{PL}{5} = \frac{12}{13};$$

hence $$PL = \frac{60}{13}; \quad \therefore PQ = \frac{120}{13}.$$

Thus $$PQ = 9\tfrac{3}{13}''.$$

EXERCISES XXXIV

1. In a circle of $1''$ radius draw the secant PCP', passing through the centre C. Cut off $CP = CP' = 1.41''$. From P and P' draw pairs of

tangents to the circle. Show that the four-sided figure formed by the tangents is very nearly a square.

2. Take a circle of 2" radius and draw the secant *PCP'* passing through the centre *C*. Cut off $CP = CP' = 4''$. Draw pairs of tangents to the circle from *P* and *P'*. Measure the sides, angles, and diagonals of the quadrilateral figure thus formed.

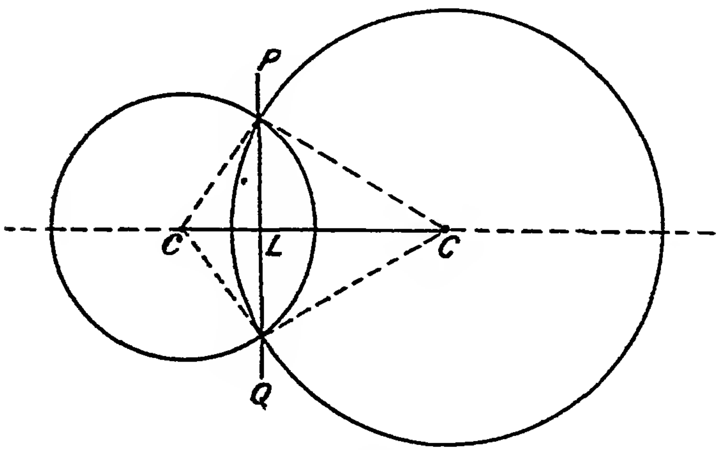
3. A pair of tangents is drawn to a circle of 3" radius from a point distant 5" from the centre. Find their lengths, and also the length of their chord of contact.

4. Draw a pair of tangents to a circle of 1" radius from a point distant 2.6" from the centre. Calculate the lengths of the tangents and their chord of contact. Also find the distance of the centre from the chord of contact.

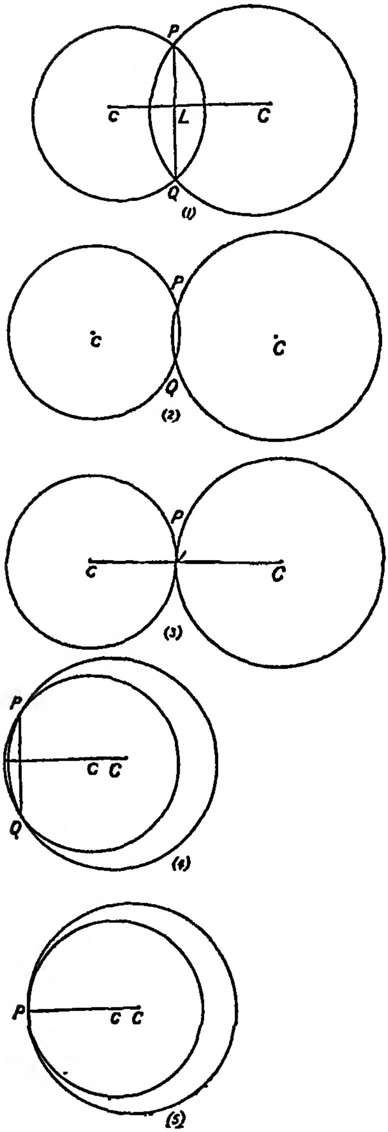
5. Describe a circle of $1\frac{3}{4}''$ radius, and from a point $6\frac{1}{4}''$ from the centre draw a pair of tangents. Find the distance of the chord of contact from the centre.

Two Touching Circles

128. Draw two circles cutting one another as in the figure.



The line *Cc*, which passes through the centres of both circles, is called the **line of centres**.



The line PQ, which joins the points of intersection, is called the **chord of intersection,** and when produced both ways it is called the **secant of intersection.** Each circle is symmetrical about a diameter, hence it follows that the whole figuıe is symmetrical about the line of centres, so that if the figure is folded about the line Cc one part exactly fits the other.

The student ought to cut out the figure and see for himself that this is so.

In the folded figure the point P will fall on the point Q; it follows therefore that—

The line of centres is the perpendicular bisector of the chord of intersection, and generally the line of centres is at right angles to the secant of intersection.

129. Draw two circles, and cut them out carefully, so that the circum-

ferences are well defined. Place one on top of the other as in (1).

Now, keeping the small circle fixed, move the larger one towards the right into the position (2). During this operation the points P and Q will be coming nearer and nearer to each other and the line of centres; and since PQ is always bisected by the line of centres, they will arrive together at that line as in (3), and coalesce into one point P.

Similarly, if the larger circle is moved towards the left into the position (4), the points P and Q will approach each other, and finally coincide, as in (5), on the line of centres.

When the points of intersection of two circles approach one another and coincide, the circles are said to touch one another.

In (3) the circles touch one another *externally*, and the centres are as far apart as possible.

In (5) the circles are touching internally, and the centres are as near each other as possible.

The point P in both cases is the point of contact.

In the above experiment we have seen that the points P and Q coincide, and become one point P on the line of centres; hence—

When two circles touch internally or externally, the point of contact lies on the line of centres.

EXERCISES

1. Describe two circles of 2″ and 3″ radius touching each other externally.

(*The centres are at the extremities of a line* 3 + 2 *inches long.*)

2 Describe two circles of $2\frac{1}{2}''$ and $1\frac{3}{4}''$ radius touching one another internally.

(*The centres are at the extremities of a line* $2\frac{1}{2} - 1\frac{3}{4}$ *inches long.*)

3. Describe a series of circles of radii $\frac{1}{4}''$, $\frac{1}{2}''$, $\frac{3}{4}''$, $1''$, $1\frac{1}{4}''$, $1\frac{1}{2}''$, $1\frac{3}{4}''$, and $2''$, having their centres on a straight line, and touching externally.

4. Describe four circles touching internally at the same point and having radii of $\frac{1}{4}''$, $\frac{1}{2}''$, $1''$, and $2''$ respectively.

5. Describe three equal circles of radius $1''$ touching one another externally.

(*The centres are the angular points of an equilateral triangle of* $2''$ *side.*)

6. Place seven pice flat on the table so that one is in the centre and the other six around it touching it externally.

Make a drawing of this arrangement, taking the diameter of a pice to be an inch.

(*Make use of the last exercise.*)

7. Describe four circles of $1''$ radius each touching two others externally; describe a fifth circle touching the four externally, and a sixth touching them internally.

(*The centres of the four are the angular points of a square of* $2''$ *side. The centre of the other two is the point of intersection of the diagonals of the square.*)

8. Describe three circles touching one another externally, and two others touching the first set internally and externally.

(*The centre of the last two is the point where the bisectors of the angles of the equilateral triangle of Ex.* 5 *meet.*)

Common Tangents to Two Circles

130. *Draw the direct common tangents to two circles.*

Let C, c be the centres. Join Cc.

Draw a circle with centre C and radius equal to the difference of the two radii.

From c draw tangents cP, cP' to this circle. Join CP, CP', cutting the circle whose centre is C in T, T'.

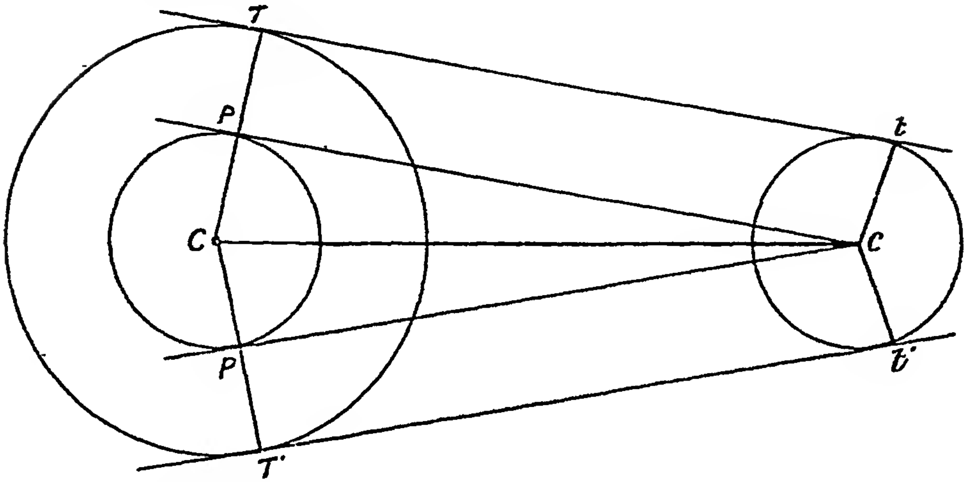
Draw the radii ct, ct' at right angles to cP, cP'.

Then Tt, $T't'$ are the direct common tangents. Since

CP is equal to the difference of the radii of the two circles, PT must be equal to the radius of the smaller circle.

The figure $TPct$ is a rectangle, for the angles at P and c are right angles, and $PT = ct$.

Thus Tt makes right angles with the radii of both circles, hence it is a common tangent.



Similarly, $T't'$ is a common tangent.

The student ought to notice that

$$Tt = cP.$$

Thus the length of the common tangent is equal to the length of the tangent drawn from the centre of the smaller circle to the circle of construction.

Ex. *The centres of two circles are $6\frac{1}{2}$ inches apart, and their radii are $3\frac{1}{2}''$ and $1''$ respectively. Draw their direct common tangents, and calculate their lengths.*

In the right-angled triangle CcP we have

$$Cc = 6\tfrac{1}{2}, \quad CP = 3\tfrac{1}{2} - 1 = 2\tfrac{1}{2};$$

therefore $cP^2 = (6\frac{1}{2})^2 - (2\frac{1}{2})^2 = 36.$

Hence $cP = 6$: therefore $Tt = 6$.

Also $T't' = Tt = 6$.

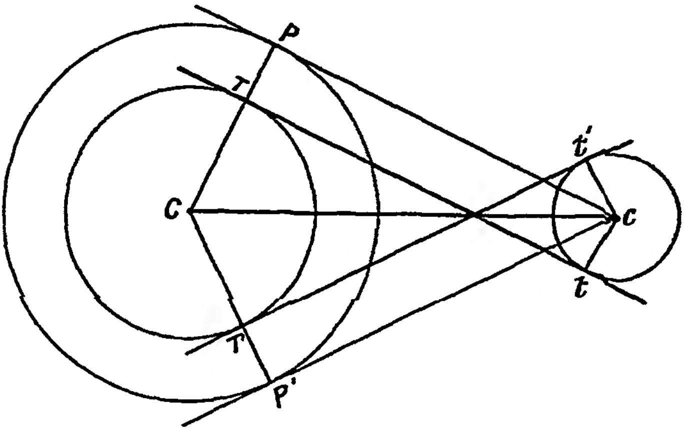
131. *Draw the transverse common tangents to two circles.*

Let C, c be the centres. Join Cc.

Draw a circle with centre C and radius equal to the sum of the two radii.

From c draw tangents cP, cP' to this circle.

Join CP, CP', cutting the circle whose centre is C in T, T'.



Draw the radii ct, ct' at right angles to cP, cP'.

Then Tt, $T't'$ are the transverse common tangents.

Since CP is equal to the sum of the radii of the two circles, PT must be equal to the radius of the smaller circle.

The figure $TPct$ is a rectangle, for the angles at P and c are right angles, and $PT = ct$. Thus Tt makes right angles with the radii of both circles, hence it is a common tangent.

Similarly, $T't'$ is a common tangent.

Here also we notice that

$$Tt = cP$$

Ex. Find the length of the transverse common tangents of the circles in the example of Art. 130. Consider the right-angled triangle CcP in the figure of Art. 131. We have

$$Cc=6\tfrac{1}{2},\ CP=3\tfrac{1}{2}+1=4\tfrac{1}{2};$$

therefore $cP^2=(6\tfrac{1}{2})^2-(4\tfrac{1}{2})^2=22$.

Hence $cP=\sqrt{(22)}=4.7$ nearly.

Thus $Tt=T't'=4.7''$.

EXERCISES XXXV

1. Describe two circles of $2\tfrac{1}{2}''$ and $1\tfrac{1}{2}''$ radius respectively, having their centres $5''$ apart. Draw their direct common tangents, and produce them to meet. Show that they intersect on the line of centres.

2. Describe two circles of $2\tfrac{1}{4}''$ and $\tfrac{3}{4}''$ radius respectively, having their centres $4\tfrac{1}{2}''$ apart. Draw their transverse common tangents, and verify that they intersect on the line of centres.

3. The centres of two circles are $5''$ apart, and their radii are $3\tfrac{1}{2}''$ and $1\tfrac{1}{2}''$ respectively. Draw their direct common tangents.

Where are the transverse common tangents?

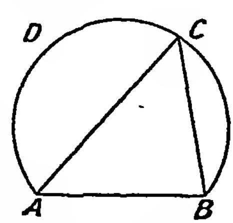
4. Draw two equal circles of $2''$ radius, the centre of each lying on the circumference of the other. Draw their direct common tangents, and find their lengths.

5. In Examples 1, 2, and 3 calculate and measure to two places of decimals the lengths of the common tangents drawn.

6. Describe two circles of $3''$ and $2''$ radius respectively, with their centres A, B at a distance of $7''$ from one another. Draw their direct common tangents meeting the line of centres at P and transverse common tangents meeting the same line in Q. Show that $\frac{PB}{PA}=\frac{QB}{QA}=\frac{2}{3}=$ratio of the radii.

Prove by means of similar triangles that this is generally true for all circles.

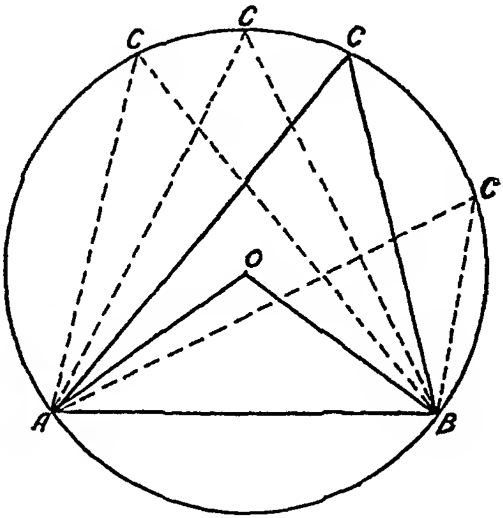
Angles in the Segment of a Circle



132. *The angle contained by two straight lines drawn from any point in the arc of a segment to the extremities of its chord is called* **an angle in the segment**; *and the segment is said to* **contain** *the angle.*

The angle ACB is said to be an angle in the segment ADB, and the segment ADB is said to contain the angle ACB.

Draw a circle ABC. Take any arc AB and join its extremities to the centre O.



Take any point C on the remaining part of the circumference and join it to A and B. Measure the angles ACB and AOB, and you will find that the angle at O is double of the angle at C, *i.e.* the angle which the arc AB subtends at the centre is double of the angle which it subtends at the remaining part of the circumference. By taking the point C in various positions you will find that the same rule holds good.

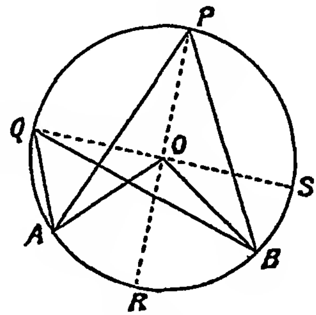
Ex. 1. Describe a circle of 2″ radius, and take in it an arc AB subtending an angle of 60° at the centre, by drawing the radii OA, OB containing this angle. Now take five points on the remaining part of the circumference, and verify by measurement that the arc AB subtends an angle of 30° at each of these points.

Ex. 2. Repeat Exercise 1 by taking arcs which subtend angles of 120°, 90°, 240°, 270°, and 300° respectively at the centre.

133. In the figure of the last article all the angles, such as C, are halves of the same angle AOB, and are therefore equal. Whence it follows that—

Angles in the same segment of a circle are equal.

We can prove this independently also.



Let APB, AQB be two angles in the same segment of a circle. Join A and B with O, the centre of the circle; and produce PO, QO to R and S.

The angle AOR is equal to the sum of the angles OPA, OAP; and these angles being equal, each of them is equal to half the angle AOR.

We have

$$\begin{aligned}\text{the angle } APB &= \text{angle } APO + \text{angle } BPO\\ &= \tfrac{1}{2}(\text{angle } AOR) + \tfrac{1}{2}(\text{angle } BOR)\\ &= \tfrac{1}{2}(\text{angle } AOB).\end{aligned}$$

$$\begin{aligned}\text{Again, the angle } AQB &= \text{angle } AQO - \text{angle } BQO\\ &= \tfrac{1}{2}(\text{angle } AOS) - \tfrac{1}{2}(\text{angle } BOS)\\ &= \tfrac{1}{2}(\text{angle } AOB).\end{aligned}$$

Therefore the angles APB and AQB are equal.

The student will notice that the angles APB, AQB stand upon the same arc AB. Hence we may say—

Angles which stand upon the same arc of a circle are equal.

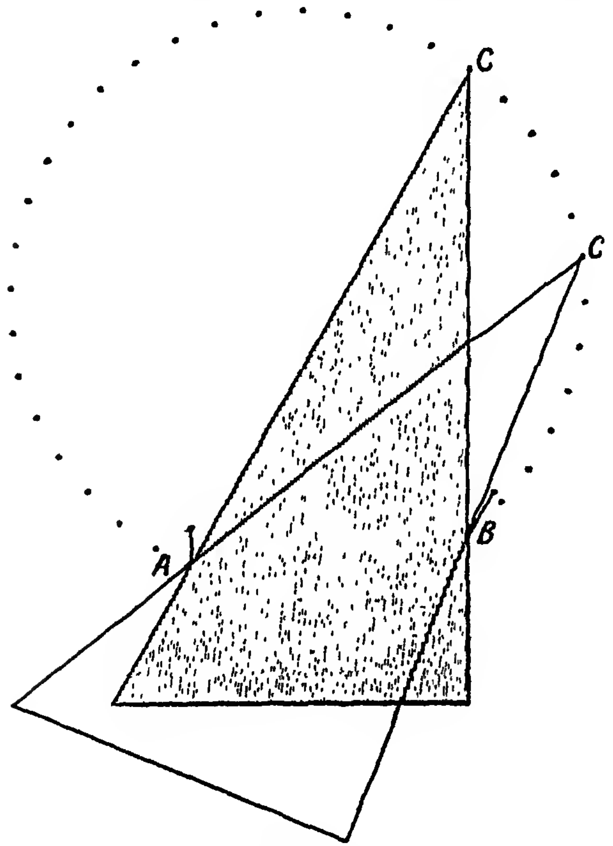
In the course of the demonstration we have seen that *the angle subtended by an arc at the centre of a circle is double of the angle subtended by the arc at the circumference.*

The converse of this is also true, viz.: *If A, B be two fixed points, and if a point C move on the same side of the line*

AB in such a manner that the angle ACB remains always the same, then the locus of C is a segment of a circle.

This may be proved by the following experiment:—

Fix two nails *A*, *B* on the table. Place your set-square

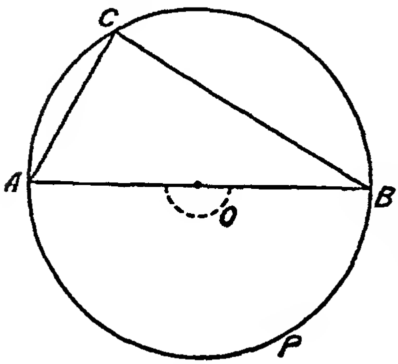


between the nails as in the figure. Now move the set-square so that the sides containing the angle of 30° constantly press against the nails. Mark the different positions of the angle *C* by means of dots. You will thus obtain a

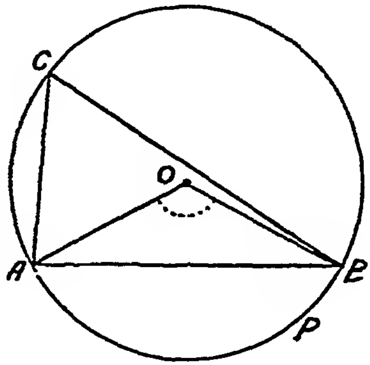
curved line of dots, which will be a segment of a circle containing an angle of 30°

Ex. Repeat the same experiment by taking the moving angle of 45°, 60°, and 90° respectively; and see that in the last case the segment described is a semicircle.

134. Consider three segments of a circle, one a semicircle, the second greater than a semicircle, and the third less than a semicircle.



(1)

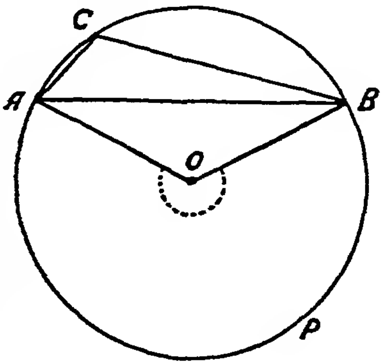


(2)

In (1) the angle subtended by the arc APB at the centre is two right angles, hence the angle C at the circumference is a right angle.

In (2) the angle subtended by APB at the centre is less than two right angles, hence the angle C at the circumference is less than a right angle.

In (3) the angle subtended by APB at the centre is greater than two right angles,



(3)

hence the angle C at the circumference is greater than a right angle.

Now in (1) C is the angle in a semicircle; in (2) it is the angle in a segment greater than a semicircle; and in (3) it is the angle in a segment less than a semicircle.

Therefore *the angle in a semicircle is a right angle; the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.*

EXERCISES

1. Take a straight line and from it cut off $AB=4''$ and $BC=1''$. On AC describe a semicircle, and draw BL at right angles to AC meeting the circumference in L. Join AL and CL.

Find by measurement the length of BL. Find also the length of BL, from the consideration, that ALC is a right-angled triangle and LB is the perpendicular from the right angle on the hypotenuse (Art. 97). Do the two results agree?

2. From any straight line cut off $AL=5''$ and $LB=3''$. On AB describe a semicircle, and draw LC at right angles to AB to meet the circumference in C. Join AC, BC. Then since the angle in a semicircle is a right angle, ACB is a right-angled triangle; therefore by Art. 97 we have

$$CL^2=AL.BL,$$
$$\therefore CL=\sqrt{(15)}.$$

Hence this construction gives us the mean proportional to two given straight lines.

Also notice that CL is the side of a square which has the same area as the rectangle whose sides are AL and BL.

3. Find by construction the side of a square which has the same area as a rectangle whose sides are $3''$ and $4''$.

4 Construct a rectangle whose sides are 4 cm. and 7 cm., and construct a square having the same area.

5 Draw a triangle with sides $3.8''$, $4''$, $3.6''$ and construct a rectangle equal to it in area (Art. 87, Ex. 1); finally make a square equal to the rectangle.

6. Draw any four-sided figure; reduce it to a triangle (Art. 89);

reduce the triangle to a rectangle (Art. 87); and finally reduce the rectangle to an equivalent square.

In the same way we can construct a square equal to any rectilineal figure.

7. Construct a triangle whose sides are 13, 14, and 15 centimetres long; find by construction the side of a square which has the same area.

8. Draw an equilateral triangle of 3″ side, and construct a square of equal area.

9. On a base 7.8 cm. long construct an isosceles triangle whose vertical angle is of 30°, and make a square equal to it in area.

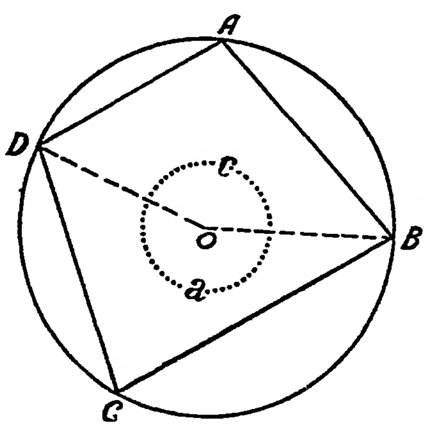
10. Taking one centimetre as unity find by construction the lines represented by:—

(i.) $\sqrt{(6)}$. (ii.) $\sqrt{(8)}$. (iii.) $\sqrt{(10)}$. (iv.) $\sqrt{(7)}$.

Cyclic Quadrilateral

135. Describe a circle, and in it draw an inscribed quadrilateral *ABCD*.

Such a quadrilateral is called a **cyclic quadrilateral.** By drawing a number of cyclic quadrilaterals and measuring their angles the student will find that the sum of a pair of opposite angles is always equal to two right angles.



We can also deduce this result from the preceding proposition.

For joining B and D to the centre we have

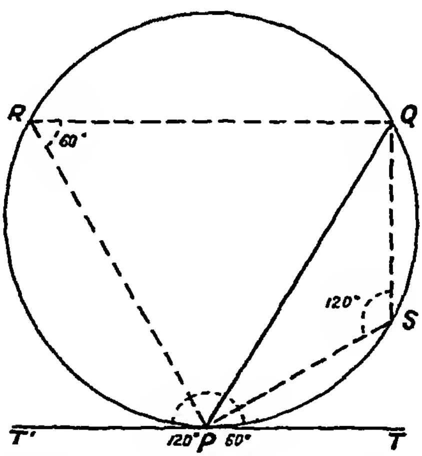
the angle marked a = twice the angle A
" " c = " " C.

Therefore the sum of the angle a and c = twice the sum of the angles A and C. But the angles a and c are together equal to four right angles. Therefore the sum of the angles A and C is equal to two right angles.

Hence *the opposite angles of a cyclic quadrilateral are supplementary*.

Angles in Alternate Segments

136. Describe a circle, and draw the tangent at any point. Through the point of contact draw a chord making an angle of 60° with the tangent.



Now measure the angles in the two segments. You will find that the angle in one segment is of 60°, and

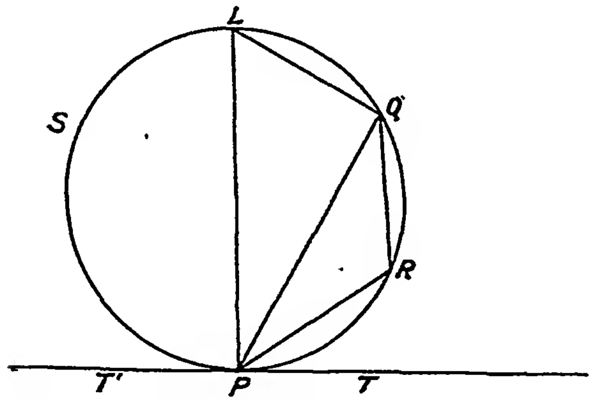
in the other of 120°. But these are the angles which the chord makes with the tangent.

Notice that the angle of 60° is contained in the segment which lies on the side of QP opposite to that on which QP makes 60° with the tangent. This fact is expressed by saying that the chord makes angles with the tangent which are equal to the angles in the alternate segments.

Ex. Repeat the experiment of this article by drawing in different circles chords which make angles of 30°, 45°, 50°, 75°, 120°, 90°, and 135° respectively with the tangent.

We shall now give a formal proof of this proposition.

Let TPT' be a tangent, and PQ the chord drawn from



the point of contact P. Draw the diameter PL which will be perpendicular to the tangent.

The angle at PQL is a right angle (Art. 134);
therefore the angle QLP = complement of LPQ.

But the angle QPT = " " LPQ.

Hence the angle QPT = the angle QLP.

Now all the angles in the segment QSP are equal to the angle QLP (Art. 133).

Therefore the angle QPT is equal to the angle in the alternate segment.

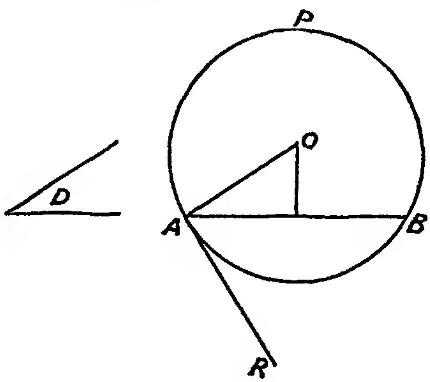
Again, the angle $QPT' =$ supplement of QPT,
and „ „ $QRP =$ „ „ QLP (Art. 135).

But $Q\hat{P}T = Q\hat{L}P$; $\therefore$ $Q\hat{P}T' = Q\hat{R}P$.

Therefore *if a straight line touch a circle, and from the point of contact a chord be drawn, the angles which this chord makes with the tangent are equal to the angles in the alternate segments.*

137. The following constructions depending on the proposition of the last article are useful :—

On a given straight line AB to describe a segment of a circle to contain a given angle D.



Make the angle BAR equal to D.

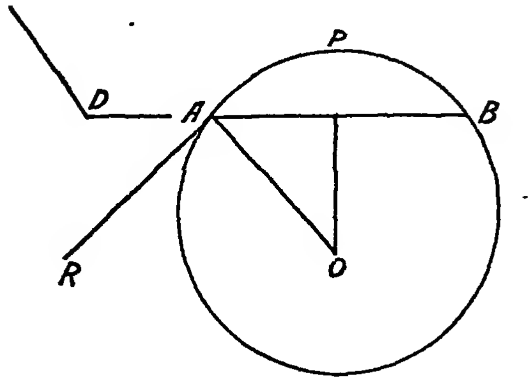
Draw AO at right angles to AR, meeting the perpendicular bisector of AB in O.

With centre O and radius OA describe a circle.

Then APB is the required segment.

Since O lies on the perpendicular bisector of AB, the

circle will pass through A and B (Art. 119); and since OAR is a right angle, AR is the tangent to the circle at A.



The angle BAR is equal to the angle in the alternate segment APB.

The two figures correspond to an acute and an obtuse angle respectively. When the given angle is a right angle the required segment is the semicircle on AB as diameter.

EXERCISES

1. On a base 2″ long describe a segment of a circle containing an angle of 60°.

2. On a base 7 cm. long describe a segment of a circle containing an angle of 75°. Draw four or five angles in this segment, and show that the lines bisecting these angles all pass through the same point. Where is this point? and why is it there?

3. Given the base of a triangle and the vertical angle, what is the locus of the vertex?

The base of a triangle is $1\frac{1}{2}$″ and the vertical angle of 30°; construct the locus of the vertex.

4. The base of a triangle is 2″, height $1\frac{1}{4}$″, and vertical angle of 60°; construct it.

On the base describe a segment containing an angle of 60°. *Draw a parallel to the base at a distance of* $1\frac{1}{4}$″, *cutting the circle in two points, which are the vertices of the required triangle. Thus there*

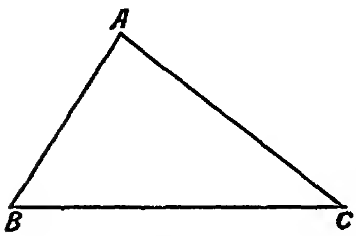
are two solutions generally, but there may be only one, or none in some cases.

5. On a base $2\frac{1}{2}''$ construct a triangle whose height is $1\frac{1}{4}''$ and vertical angle of 90°.

6. On a base $3''$ long is it possible to construct a triangle whose height is $2.8''$ and vertical angle of 60°?

7. Construct a triangle whose sides are $2''$, $2\frac{1}{2}''$, and $2\frac{3}{4}''$ respectively. On each side describe a segment of a circle lying inwards and containing an angle equal to the supplement of the opposite angle of the triangle. The three segments ought to pass through the same point.

138. *In a given circle inscribe a triangle equiangular to a given triangle.*

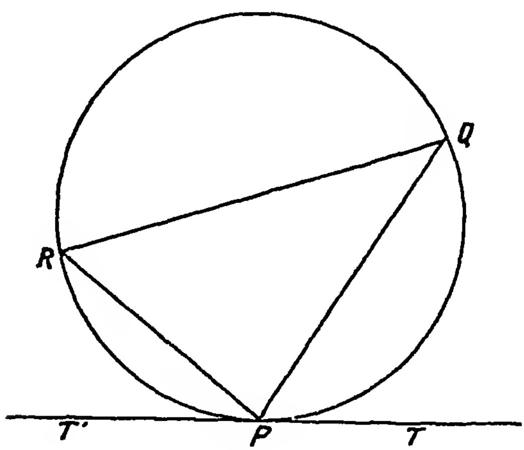


Let ABC be the given triangle.

Draw a tangent TPT' to the circle.

Make the angles TPQ, $T'PR$ equal to the angles B and C of the triangle.

Then PQR is the required triangle.



For the angle R = angle QPT in the alternate segment
= angle B.

Similarly the angle Q = angle C.

The remaining angles of the two triangles are also equal.

EXERCISES

1. On a base 1″ long construct a triangle whose angles at the base are of 36° and 45° respectively. Inscribe a triangle equiangular to this in a circle of $1\frac{3}{4}$″ radius.

2. In a circle of 3″ radius inscribe a triangle, two of whose angles are of 40° and 60° respectively.

3. In a circle of 2″ radius inscribe an equilateral triangle.

4. In a circle of $2\frac{3}{4}$″ radius inscribe an isosceles triangle whose vertical angle is of 30°.

5. In a circle of 5 cm. radius inscribe an isosceles triangle whose vertical angle is of 36°.

6. In a circle of 6 cm. radius inscribe an isosceles triangle whose angles at the base are each double of the vertical angle. Draw the bisectors of the base angles, and join the points in which these lines intersect the circle to the angular points of the triangle. Show that the five-sided figure thus formed is the inscribed regular pentagon of the circle.

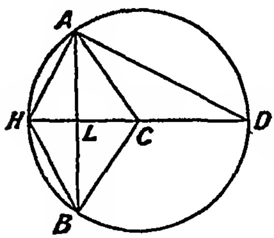
CHAPTER XIII

CIRCLE—*continued*

139. IN this chapter we shall consider certain constructions and simple calculations connected with the circle.

140. Let AB be a chord of a circle, HD the diameter which bisects the chord at right angles in L, and C the centre of the circle; then HL is called the height of the arc AHB.

By the method of superposition it can easily be shown that HL bisects the arc AHB, hence AH, or HB, is the chord of half the arc.



In the figure the angle HAD is a right angle (Art. 134), and AL is the perpendicular drawn from the right angle of a right-angled triangle upon the hypotenuse; therefore the sides, the segments of the base, and the perpendicular are connected with each other by the relations established in Art. 97.

EX. *The chord of an arc is 8 ft., and the height of the arc is 2 ft.; find the chord of half the arc and the diameter of the circle.*

By Art. 97 we have

$$AL^2 = HL \times LD,$$

$$16 = 2LD\text{; hence } LD = 8.$$

Thus the diameter is 10 ft.

Again, by the same Article, we have

$$HA^2 = HL \times HD = 2 \times 10.$$

Therefore $HA = 2\sqrt{5} = 4.47$ ft. nearly.

Thus the chord of half the arc is 4 ft. 6 in. nearly.

EXERCISES XXXVI

In the following examples when the answers are not exact they must be found correct to two places of decimals.

1. The chord of an arc is 12 ft., and the height of the arc is 3 ft.; find the diameter of the circle.

2. The height of an arc is 8 in., and the chord of half the arc is 1 ft.; find the diameter of the circle.

3. The chord of an arc is 3 ft. 4 in., and the chord of half the arc is 2 ft. 1 in.; find the height of the arc.

4. The height of an arc is 1 ft., and the chord of half the arc is 1 ft. 8 in.; find the chord of the whole arc.

5. The height of an arc is 4 ft., and the diameter of the circle is 20 ft.; find the chord of the arc.

6. In a circle whose radius is 1 ft. 3 in. a chord of 1 ft. 6 in. is placed; find the heights of the two arcs into which it divides the circle.

7. A chord of 4 ft. can be placed at a perpendicular distance of 7 in. from the centre of a circle; find the radius of the circle.

8. In a circle whose radius is 1 ft. 1 in. a chord is drawn at a distance of 5 in. from the centre; find the length of the chord.

9. A chord is drawn bisecting at right angles the radius of a circle; show that the chord of half the lesser arc is equal to the radius of the circle.

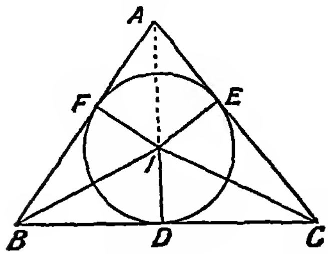
10. In a circle whose diameter is 1 ft. 3 in. the chord of an arc is 3 in.; find the chord of an arc twice as great.

11. The diameter of a circle is 100 ft.; parallel chords of lengths 60 and 80 ft. respectively are drawn and their extremities joined to form a trapezoid; find its area.

12. In a circle whose radius is 1 ft. two arcs have a common chord, and the height of one is twice that of the other; find the length of the chord.

Inscribed Circle

141. DEFINITIONS.—The circle inscribed in a rectilineal figure is called its **incircle**; and the centre and radius of this circle are called **the incentre** and **the inradius** respectively.



To inscribe a circle in a given triangle.

Bisect any two angles B, C of the triangle. The point of intersection I of the bisectors will be the centre of the required circle.

Draw ID, IE, IF perpendiculars on the sides.

Every point on BI is equidistant from AB and CB (Art. 112). Therefore

$$ID = IF.$$

Again, every point on CI is equidistant from CA and CB (Art. 112). Therefore

$$ID = IE.$$

Hence $$ID = IE = IF.$$

Therefore a circle described with I as centre and ID as radius will pass through E and F; and since the angles at D, E, F are right angles, it will touch the sides of the triangle (Art. 124).

If AI be joined, it can be proved that it bisects the angle A. Hence *the three bisectors of the angles of a triangle meet at its incentre.*

To find the inradius of a triangle.

We have

$$\textit{area } ABC = \textit{area } IBC + \textit{area } ICA + \textit{area } IAB$$
$$= \tfrac{1}{2}(\textit{inradius} \times BC) + \tfrac{1}{2}(\textit{inradius} \times CA) + \tfrac{1}{2}(\textit{inradius} \times AB)$$

$$= inradius \times \tfrac{1}{2}(BC + CA + AB)$$
$$= inradius \times semiperimeter.$$

Hence $$inradius = \frac{area\ of\ triangle}{semiperimeter}.$$

Ex. *The sides of a triangle are* 13, 14, *and* 15; *find the radius of a circle inscribed in it.*

The semiperimeter $=\frac{1}{2}(13+14+15)=21$, and the
$$\text{area} = \sqrt{21\times 8\times 7\times 6} = 84;$$
therefore the inradius $=\frac{84}{21}=4.$

Thus the radius of the inscribed circle is 4.

EXERCISES XXXVII

1. The radius of a circle is 3 in.; find the length of a tangent drawn to it from a point whose distance from the centre is 5 in.

2. The radius of a circle is 1 ft. 1 in., and the length of the chord of contact of a pair of tangents drawn to it from an external point is 2 ft.; find the distance of the point from the centre of the circle.

3. A circle whose radius is 4 ft. 2 in. passes through the centre of another, whose radius is 5 ft.; find the length of the line joining their points of intersection.

4. The base of an isosceles triangle is 14, and each side is 25; find the inradius.

Find correct to two places of decimals the inradii of triangles having the following sides:—

5. 5, 6, 7.	6. 8, 10, 11.
7. 1, 1 1, 1.2.	8. 41, 40, 11.
9. 35, 21, 16.	10. 9, 12, 20.

11. Prove that the inradius of an equilateral triangle is equal in length to one-third of its height.

12. In a given square inscribe a circle, and show that the diameter of the circle is equal to a side of the square.

13. Inscribe a circle in a given rhombus, and show that its diameter is equal to the height of the rhombus.

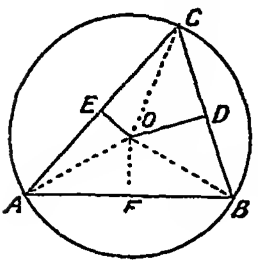
14. The radius of the inscribed circle of a rhombus is 4 in., and its area is half a square foot; find the length of a side.

15. The perimeter of an equilateral triangle is 1.732 ft.; show that the inradius is 2 in. nearly.

16. The sides of a triangle are proportional to the numbers 41, 41, 80; and the inradius is 40 yds.; find the area.

Circumscribed Circle

142. DEFINITIONS.—The circle described about a rectilineal figure is called its **circumscribing circle**, or **circumcircle**. The centre and radius of this circle are called **the circumcentre** and **the circumradius** respectively.



To describe a circle about a given triangle ABC.

Bisect any two sides BC, CA; and through the points of bisection draw DO, EO at right angles to the sides and meeting in O. Then O will be the centre of the required circle.

Every point on DO is equidistant from B and C (Art. 111). Therefore

$$OB = OC$$

Every point on EO is equidistant from A and C. Therefore

$$OA = OC.$$

Hence $$OA = OB = OC.$$

Therefore a circle described with O as centre and OA as radius will pass through the three angular points.

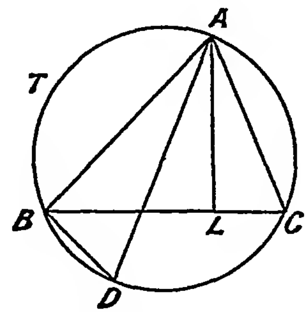
Since OAB is an isosceles triangle, if OF be drawn perpendicular to AB, it will bisect AB. Hence we see that *the straight lines which bisect at right angles the three sides of a triangle meet in a point, which is the circumcentre of the triangle.*

To find the circumradius of a triangle ABC.

Draw AL at right angles to BC. Through A draw the diameter AD. Join BD.

In the two triangles ABD, ALC the angles at C and

D, which stand on the same arc ATB, are equal. The angle ABD in a semicircle is a right angle, and therefore equal to the angle ALC. Hence the two triangles are similar, and we have



$$AD : AB :: AC : AL.$$

Therefore

$$AD \times AL = AB \times AC,$$

whence

$$AD \times AL \times BC = AB \times AC \times BC.$$

But $AD = 2$(circumradius), and $AL \times BC = 2$(area); therefore 4(circumradius) × (area) $= AB \times AC \times BC$:

hence we get $\textit{circumradius} = \dfrac{\textit{continued product of sides}}{4(\textit{area})}$.

EXERCISES XXXVIII

1. Show that the centre of a circle described about a right-angled triangle is at the middle point of the hypotenuse.

2. Show that the circumradius of an equilateral triangle is two-thirds of its height.

3. Describe a circle about a given square, and show that its diameter is equal to the diagonal of the square.

4. Describe a circle about a given rectangle.

5. Find the radius of a circle described about a rectangle whose sides are 180 and 112 ft. respectively.

Find the circumradii of triangles having the following sides:—

6. 52, 56, 60. **7.** 18, 41, 41.

8. 51, 52, 53. **9.** 204, 225, 231.

10. Each of the angles at the base of an isosceles triangle is of 30°; show that the circumradius is equal to one of the sides.

11. The sides of a triangle are 7, 8, and 9 yds.; find correct to the hundredth part of an inch the length of the circumradius.

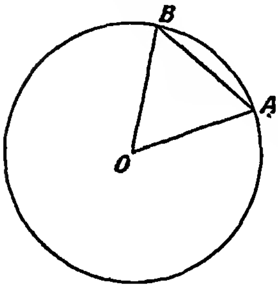
12. An equilateral triangle and a square have the same perimeter; show that their circumradii are in the ratio of $4\sqrt{2} : 3\sqrt{3}$.

13. Each side of an isosceles triangle is double of the base; compare the radii of the inscribed and circumscribed circles.

Inscribed and Circumscribed Hexagon

143. *To inscribe a regular hexagon in a circle.*

Suppose two radii OA, OB are drawn containing an angle of 60°. Join AB. The triangle OAB is evidently equilateral, therefore the chord AB is equal to the radius of the circle.



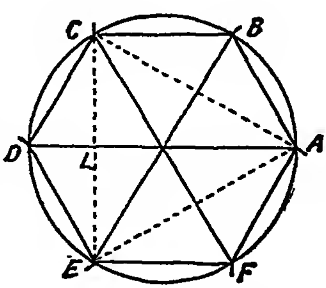
Now the angular space round O is of 360°, therefore six angles each equal to AOB will completely fill it up; hence six chords each equal to AB can be placed one after another in the circumference, so that they just go round.

Thus to inscribe a hexagon in a circle we must place in its circumference six chords each equal to the radius.

The student will notice that if we join the points A, C, E, we obtain the inscribed equilateral triangle.

Ex. 1. *The side of a hexagon being given, find its area.*

The hexagon in the figure is made up of six equilateral triangles, which have a common vertex at the centre of the circle. Hence *the area of a regular hexagon is equal to six times the area of an equilateral triangle described on its side.*



Ex. 2. *Find the side of an equilateral triangle inscribed in a circle.*

In the figure, CE is double of CL, therefore *the side of an equilateral triangle inscribed in a circle is equal to twice the height of an equilateral triangle described upon the radius.*

Ex. 3. *Find the area of an equilateral triangle inscribed in a circle.*

The triangle AEC is made up of six smaller triangles, each of which is equal to half the equilateral triangle described upon the radius. Hence *the area of an equilateral triangle inscribed in a circle is equal to three times the area of the equilateral triangle described upon its radius.*

Ex. 4. *Find the area of an equilateral triangle described about a given circle.*

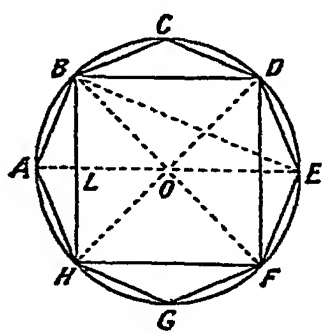
If we draw tangents at the angular points of a regular inscribed figure, we obtain a regular figure of the same number of sides described about the circle. In the present case the tangents at the points A, E, C will form an equilateral triangle whose area is equal to four times the area of the triangle AEC.

Octagons

144. *To inscribe a regular octagon in a given circle.*

Draw two diameters BOF, HOD at right angles to one another. By joining their extremities we get the inscribed square.

Draw the radius OA bisecting the angle BOH, then A will bisect the arc BH; therefore the arc AB will be an eighth part of the circumference. Hence the chord AB is equal to a side of the required octagon.



In the same way, when a regular figure having any number of sides has been inscribed in a circle, another figure having twice as many sides can be inscribed by simply bisecting the arcs.

Ex. *Find the side of a regular octagon inscribed in a circle whose radius is unity; and calculate its area.*

In the right-angled triangle BOH the hypotenuse BH is equal to $\sqrt{2}$. We thus know the chord of the arc BAH, and we are required to find the chord of half the arc.

From the right-angled triangle ABE we have

$$AB^2 = AL \times AE.$$

Now $$OL = BL = \tfrac{1}{2}BH = \frac{\sqrt{2}}{2};$$

therefore $$AL = 1 - \frac{\sqrt{2}}{2}; \text{ also } AE = 2.$$

Hence $$AB^2 = 2 - \sqrt{2}.$$

Thus the side of the inscribed octagon is equal to the square root of $2 - \sqrt{2}$.

The area of the octagon is four times the area of the quadrilateral $OBAH$ in which the diagonals intersect at right angles. By Ex. 3 of Art. 103, the area of the quadrilateral $OBAH$ is equal to $\frac{1}{2}(BH \times OA)$, or $\frac{\sqrt{2}}{2}$. Hence the area of the octagon is equal to $2\sqrt{2}$.

EXERCISES XXXIX

1. Find the perimeter of a regular hexagon which is inscribed in a circle whose diameter is 4 in.

2. One side of a regular hexagon is 10 ft.; find the area.

3. Find the perimeter of an equilateral triangle inscribed in a circle whose radius is 1 yd.

4. Find the area of an equilateral triangle described about a circle whose diameter is 1 yd.

5. About a circle 2 in. in diameter describe a regular hexagon, and find its area correct to two places of decimals.

6. Compare the areas of the inscribed and circumscribed regular hexagons of a given circle.

7. In a circle whose radius is 1 ft. a regular dodecagon is inscribed; find correct to the hundredth part of an inch the length of a side.

8. Find the area of a regular dodecagon inscribed in a circle whose radius is 1 yd.

9. Compare the area of a regular dodecagon with that of a regular hexagon inscribed in the same circle.

10. Find the area of a regular octagon inscribed in a circle whose radius is 20 ft.

11. Each side of an octagonal field is 100 yds.; find its price at Rs.225 per acre.

12. A hexagonal field contains 1 acre; find to the nearest inch the length of a side.

13. Each angle of a regular hexagon is joined with the alternate angle, and the joining lines by their intersections form another regular hexagon; compare the areas of the two hexagons.

14. On each side of a square, equilateral triangles are described all lying inwards, and the vertices of these triangles are joined to form another square. If each side of the original square be 10 ft., find to the nearest square inch the area of the smaller square.

145. **The circumference of a circle.**—Since all circles are similar figures, the ratio of the circumference to the diameter must be the same in every case.

This ratio is usually represented by the Greek letter π, so that in every circle

$$\textit{circumference} = \textit{diameter} \times \pi.$$

The value of π cannot be found exactly, but it has been calculated to a very large number of places of decimals. For our purpose it will be sufficient to take

$$\pi = 3.1416;$$

or, when a lesser degree of accuracy is required,

$$\pi = \frac{22}{7}.$$

Ex. 1. *The diameter of a circle is 5 ft.; find the circumference correct to three places of decimals.*

The circumference $= 5 \times 3.1416$
$= 15.708.$

Ex. 2. *The radius of a wheel is 2 ft. 4 in.; how many revolutions will it make in going a mile?*

Here the diameter is 56 in., and if we take $\pi = \frac{22}{7}$ we have

$$\text{circumference of wheel} = 56 \times \frac{22}{7} = 176 \text{ in.}$$

Dividing the number of inches in a mile by 176, we shall get the number of revolutions $= \dfrac{5280 \times 12}{176}$, or 360.

EXERCISES XL

Taking $\pi=3.1416$, find to three places of decimals the circumferences of circles whose diameters are—

1. 12 in.	2. 31 ft. 10 in
3. 2 yds. 1 ft. 8 in.	4. 5 yds. 3 in.
5. 10,000 yds.	6. 9 yds. 1 ft.

Taking $\pi=\frac{22}{7}$, find to the nearest inch the diameters of circles whose circumferences are—

7. 8 ft.	8. 5 ft. 6 in.
9. 121 ft. 11 in.	10. 11 yds. 2 ft. 9 in.

Taking $\pi=\frac{22}{7}$, find to the nearest inch the circumferences of circles whose diameters are—

11. 28 ft. 7 in.	12. 3 ft. 2 in.
13. 5 ft. 3 in.	14. 2 yds. 2 ft. 2 in.

Taking $\pi=3.1416$, find correct to two places of decimals the radii of circles whose circumferences are—

15. 100 ft.	16. 1 mile.

17. A carriage wheel makes 1000 revolutions in going over a distance of 3 miles; find its diameter correct to the tenth part of an inch.

18. A garden roller is 4 ft. 8 in. broad, and has a diameter of 4 ft. 8 in.; how many square feet of grass does it pass over in making 100 revolutions?

19. Find the difference between the perimeter of a circle whose radius is 1,000,000 and that of the regular inscribed dodecagon. ($\pi=3.141593$.)

20. Find the area of a regular octagon inscribed in a circle whose circumference is 1420 yds. ($\pi=355/113$.)

21. Find to the nearest inch the circumradius of an equilateral triangle whose area is 4 sq. ft.

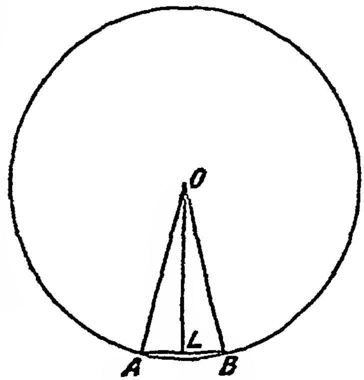
146. **Area of a circle.**—Draw two radii OA, OB containing a very small angle, and draw OL perpendicular to the chord AB.

The area of the triangle $OAB=\frac{1}{2}(OL\times AB)$.

Now when the angle AOB is very small, the perpen-

dicular OL is nearly equal to the radius; the chord AB is nearly equal to its arc; and the triangle OAB is very approximately equal to the sector OAB. Hence we have the area of the sector

$$OAB = \tfrac{1}{2}(\text{radius} \times \text{arc } AB).$$



Again, the whole circle can be divided into a large number of small sectors, and the area of the circle will be equal to the sum of the areas of these sectors; therefore

the area of the circle $= \tfrac{1}{2}$(*radius* $\times$ *circumference*).

But the circumference is equal to (2 radius) $\times \pi$, hence we get *area of circle* $=$ (*radius*)$^2 \times \pi$.

Ex. *The radius of a circle is* 10 *ft.; find its area.*
The area $=(10)^2 \times \pi = 100 \times 3.1416$
$= 314.16$ sq. ft.

147. **Area of a sector.**—We have seen that the area of a sector is proportional to its arc, and therefore proportional to the angle subtended by its arc at the centre of the circle; hence we have the following proportion:—

area of sector : *area of circle* : : *degrees in angle of sector* : 360°.

Ex. 1. *The angle of a sector is* 22° 30′, *and the radius is* 10 *ft.; find the area of the sector.*

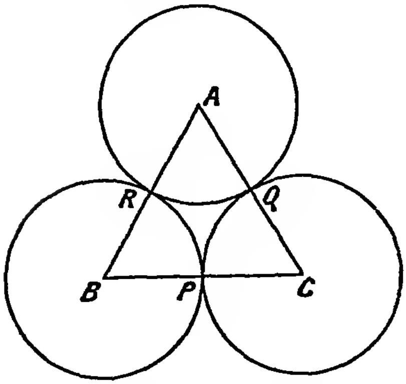
We have area of sector : $(10)^2 \times \pi : : 22\tfrac{1}{2} : 360$;

therefore area of sector $= \dfrac{314.16 \times 45}{2 \times 360} = \dfrac{314.16}{16}$
$= 19.64$ sq. ft. nearly.

Ex. 2. *Three equal circles, whose radii are each* 10 *ft., touch each other; find the area included between them.*

The curvilinear figure PQR = triangle ABC − the sectors ARQ, BRP, CPQ.

The triangle ABC is equilateral, and since a side is 20 ft., its area



$$=\tfrac{1}{2}\left(20\times 20\times\frac{\sqrt{3}}{2}\right)\text{ sq. ft.}$$
$$=100\times\sqrt{3}\text{ sq. ft.}$$
$$=173.2\text{ sq. ft. nearly.}$$

Again, the angle of the sector ARQ is of 60°, therefore its area

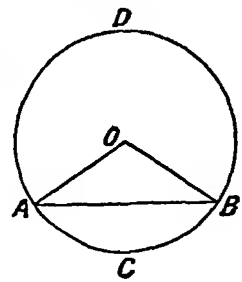
$$=\frac{60}{360}\times(10)^2\times 3.1416$$
$$=\tfrac{1}{6}\times 314.16\text{ sq. ft.}$$

Hence the sum of the areas of the three sectors

$$=\tfrac{1}{2}\times 314.16$$
$$=157.08\text{ sq. ft.}$$

Thus the area $PQR = 173.2 - 157.08 = 16$ sq. ft. nearly.

148. **The area of a segment.**—According as the segment is greater or less than a semicircle, it may be regarded as the sum, or the difference, of a sector and an isosceles triangle.



Thus *the segment ADB*

= *sector OADB + triangle OAB;*

and *the segment ACB*

= *sector OACB − triangle OAB.*

149. **Conical tent.**—The area of the canvas required for a conical tent may be found by cutting the tent along its slant length, and spreading it out into the sector of a circle.

The slant length of the tent is the radius of the sector, and the circumference of the base of the tent is its arc; therefore

the area of canvas = $\tfrac{1}{2}$(*slant side*) × (*circumference of base*).

EXERCISES XLI

In the following examples $\pi = 3.1416$, except where otherwise stated.

1. The radius of a circle is 19 ft.; find the area correct to a square inch.

2. The radius of a circle is 125 ft.; find the area correct to a square foot.

3. The circumference of a circle is 100 ft.; find the area.

4. Find the radius of a circle whose area is 31,416 sq. links.

5. A circular field contains 1 acre; find its diameter.

6. The radii of two concentric circles are 5 ft. and 4 ft. respectively; find the area of the circular ring enclosed by them.

7. A circular garden contains 2 acres; find the area of a road 3 yds. wide which completely surrounds it.

8. The area of a uniform road round a circular garden is 1 acre, and the area of the garden is 5 acres; find the breadth of the road.

9. The area of a semicircular table is 18 sq. ft.; find its perimeter to the nearest inch.

10. The side of an equilateral triangle is 8 ft.; a circle is inscribed in it, and another described about it; find the area of the circular ring.

11. An equilateral triangle and a regular hexagon have the same perimeter. Prove that the areas of their incircles are as 4 : 9.

12. The radius of a sector is 10 ft., and the length of its arc is 5 ft.; find the area of the sector.

13. The radius of a sector is 29 ft. 6 in., and its arc subtends an angle of 120° at the centre; find the area of the sector.

14. The area of a sector is one-tenth that of the complete circle; find the angle of the sector.

15. With the corners of a square as centres, four equal circles are described, each touching two others; if the side of the square be 2 in., find the area of the curvilinear space enclosed by the circles.

16. The radius and chord of a sector are each 2 ft.; find its area.

17. In a circle of diameter 10 ft. a cord is drawn bisecting a radius at right angles. Find the area of both the segments into which it divides the circle.

18. Find the area of the smaller segment cut off from a circle of radius 12 ft. by a side of the regular inscribed octagon.

19. How many yards of canvas which is 54 in. wide will be required

to make a conical tent which is 12 ft. in perpendicular height, with a radius of 9 ft ? ($\pi=22/7$.)

20. The diameter of a halfpenny is 1 in. ; 6 of these coins are placed with their centres at the corners of a regular hexagon which is of such a size that adjacent coins touch ; find the area of that portion of the hexagon which is uncovered by the coins.

MISCELLANEOUS EXAMPLES

1. Draw two circles, with radii 1.5″ and 2″, touching one another externally. (Art. 129, Ex. 1.)

2. Draw two circles, with radii 2″ and 3″, touching one another internally. (Art. 129, Ex. 2.)

3. Draw an equilateral triangle of 3″ side; with the angular points as centres and radii of 1.5″ describe circles.

4. In a circle of 2″ radius inscribe a regular hexagon; with the angular points as centres and radii of 1″ describe arcs of circles lying within the hexagon.

5. Draw an equilateral triangle of 3″ side, and trisect each side; with the angular points as centres and radii of 1″ describe portions of circles lying outside the triangle; on the middle segment of each side describe semicircles lying within the triangle.

6. Draw a triangle with sides 2″, $2\frac{1}{4}$″, $2\frac{3}{4}$″, and in it inscribe a circle

7. Draw three circles of $1\frac{3}{4}$″ radius, each one touching the other two. (Art. 129, Ex. 5.)

8. Construct a rhombus having its sides 3 cm. long and one angle 60°; within it inscribe a circle.

9. Describe two circles, with radii $1\frac{1}{2}$″ and $\frac{3}{4}$″ and centres 3″ apart; draw their direct common tangents.

10. Describe two circles with radii 32 mm. and 16 mm. and centres 60 mm. apart; draw their transverse common tangents.

11. Draw two circles, with radii $2\frac{1}{4}$″ and $1\frac{1}{4}$″ and centres $3\frac{1}{2}$″ apart; draw their two direct common tangents and their single transverse common tangent, and show that the transverse tangent bisects each of the other two.

12. Draw a circle of 4 cm radius and place in it a chord 5 cm. long; measure the angles in the two segments made by the chord, and show that they are supplementary.

13. Draw a circle of 2.3″ radius and place in it a chord 2.3″ long. Show that the greater segment contains an angle of 30°, and measure the angle in the smaller segment.

14. From a circle of $1.5''$ radius cut off a segment containing an angle of $30°$.

15. On a base $2''$ long describe a segment of a circle containing an angle of $45°$.

16. On a base 5 cm. long describe a segment of a circle containing an angle of $150°$.

17. Construct a triangle whose base is $1.5''$, height $2''$ and vertical angle of $30°$.

18. With a hypotenuse $3''$ long construct a right-angled triangle, the perpendicular from the right angle on the hypotenuse being $1''$ in length.

19. Find a mean proportional between two lines which are $1.5''$ and $2''$ long.

20. Draw two concentric circles of $3''$ and $2''$ radius; draw a number of chords of the outer circle which touch the inner, and show that they are all equal.

21 Within a rhombus of which the diagonals are $3''$ and $2.5''$ inscribe a circle.

22. Within a circle of $1.5''$ radius inscribe a triangle two of whose angles are of $57°$ and $63°$ respectively.

23. Draw a triangle with sides $1\frac{3}{4}''$, $2''$, $2\frac{1}{4}''$, and inscribe a similar triangle in a circle of $2''$ radius.

24. Describe three circles having radii of $1''$, $1\frac{1}{2}''$, and $2''$ respectively, each circle touching the other two.

(The angular points of a triangle whose sides are $(1+1\frac{1}{2})$, $(1\frac{1}{2}+2)$, $(2+1)$ inches respectively are the centres of the three circles.)

25. Describe three circles having radii of 3 cm., 3 5 cm., and 4 cm. respectively, each touching the other two.

26. Draw an equilateral triangle of 6 cm. side; trisect each side; with the angular points as centres, and radii of 4 cm., describe portions of circles lying outside the triangle, and on the middle segments of the sides describe semicircles lying within the triangle.

27. On a chord AB, $3''$ long, draw the segment of a circle to contain an angle of $54°$. Produce AB to P, making BP equal to $1.6''$. Through P draw a tangent to the segment.

28. Draw the segment of a circle to contain an angle of $45°$, the chord measuring $1.6''$. Measure the length of the radius of the circle.

29. Draw a triangle ABC with sides $3''$, $3.25''$, $3.75''$, and through A, B, C draw perpendiculars to the opposite sides meeting one another in P. On AP, BP, and CP as diameters describe circles.

30. In a circle of two inch radius inscribe a regular hexagon

ABCDEF; join *FC. CE, EF, FD, DB, BF*; these lines will enclose another regular hexagon.

31. Determine by construction a mean proportional to two straight lines 1.6" and 2.5" long, and a fourth proportional to the same two lines and the mean proportional.

32. With a radius 2.7" describe a circle, and divide it into five sectors each having the same area.

33. With a radius of 1" describe a circle, and about it describe six equal circles, each touching two of the others and the first circle.

34. Describe a hexagon of 1.8" side.

35. Describe an octagon of 1.3" side, and in it inscribe a circle.

36. Describe a hexagon of 1.6" side, and on each side of the hexagon describe semicircles lying externally.

37. On a base 1" long describe a hexagon, and reduce it to a triangle of equal area.

Construct a square equal in area to the triangle.

38. On a base 2.7" long construct an isosceles triangle of which the angles at the base are together equal to the angle at the vertex.

39. Construct an isosceles triangle of 1" base and 4" side, and find a mean proportional between the base and side.

40. Construct a square of 6 inches area.

41. Taking 1" as the unit, find two straight lines represented by the square root of 2 and the square root of 12.

42. Draw the inscribed circle of the triangle whose sides are 5 cm., 9 cm., and 7 cm. respectively.

43. Describe an equilateral triangle *ABC* of 3" side; let the bisectors of the angles meet in *O*; draw the inscribed circles of the triangles *OBC, OCA, OAB*.

44. Describe an equilateral triangle *ABC* of 2.5" side; draw *AP, BQ, CR* perpendiculars to the sides, meeting one another in *O*; inscribe circles in the quadrilaterals *OPQC, ORQA, ORPB*.

(*N.B.*—The centre of the inscribed circle will lie on lines bisecting the angles of the quadrilateral.)

45. Take two points 1.4" apart, and describe a circle of 2.4" radius passing through them.

46. Draw *OBC* a quadrant of a circle whose centre is *O* and radius 2": draw *OD* the bisector of the angle *BOC* meeting the circumference in *D*; erect *DE* perpendicular to *OD* meeting *OB* produced in *E*; cut off *EF* equal to *ED*; erect *FP* perpendicular to *OB* meeting *OD* in *P*. with *P* as centre and *PD* as radius describe a circle.

This is the inscribed circle of the quadrant *OBC*.

47. Draw a circle of 2″ radius, and draw two perpendicular diameters dividing it into four quadrants; in each quadrant inscribe a circle.

48. In a circle of 4 cm. radius draw a chord so that the angle which it subtends at the circumference may be of 30°.

49. Draw two tangents to a circle of 2″ radius, so that they may contain an angle of 45°.

50. Describe a circle with a radius of 4 cm. Inscribe in it a regular hexagon, and circumscribe about it a regular octagon.

51. What is the unit of length when a mile is represented by 80?

52. If a marla contains 272.25 sq. ft., convert 363 ghumaons into acres. (One ghumaon = 160 marlas)

53. The sides of a rectangle are 2 yds. 2 ft., and 41 yds. 2 ft. respectively; how many square feet does it contain?

54. The areas of four fields being 100, 300, 500, and 700 sq. ft. respectively, find the side of a square field which is equal in area to the sum of the four.

55. The sides of a rectangle are 352 and 114 ft. respectively; find the diagonal.

56. The diagonal of a rectangle is 290 yds., and the breadth is 34 yds.; find the area.

57. The diagonal of a square is 79 yds.; find the area.

58. The diagonal of a rectangle is 303 yds. 1 ft., and the sides have the ratio of 33 : 56; find the area.

59. The cost of putting a fence round a square field at 8p. per foot amounts to Rs.9, 8a.; find the cost of turfing it at 3p. per square yard.

60. The length of a rectangular garden is to its breadth as 4 to 3; one-fifth of the garden is planted with mango trees, and the remainder, 240 sq. chains, grows wheat. Find the length of the garden in yards.

61. The sides of a table are 9 ft. and 5 ft.; find how much border must be cut off all round that the area left may be 32 sq. ft.

62 The carpet in a room 18 ft. broad at Rs.6, 8a. a square yard costs Rs.325; find the length of the room.

63. A room is 17 ft. $8\frac{1}{2}$ in. long, 15 ft. $9\frac{1}{2}$ in. wide, and 15 ft. high; find the area of the four walls, and the cost of whitewashing them at 1a. 6p. per 100 sq. ft.

64. The cost of carpeting a room whose length is twice its breadth, at Rs.3, 2a. 6p. a square yard, is Rs.227, 4a., and the whitewashing of the walls at 2a. per 100 sq. ft. is Rs.2, 2a. 6.72p.; find the height of the room.

65. The base of a parallelogram is 35 ft., and the height is 23 ft.; find the area.

66. Two sides of a parallelogram are 15 and 16 respectively, and the included angle is of 30°; find the area.

67. Six equilateral triangles are placed, close to one another, so as to form a parallelogram; show that the greater height of this parallelogram is three times the height of one of the equilateral triangles.

68. Each of the equal sides of an isosceles triangle is 349 ft., and the height is 299 ft.; find the base.

69. The base of an isosceles triangle is 84 ft., and the height is 40 ft.; find the length of one of the equal sides.

70. The base of a triangle is 88 yds.; find the height in order that the area may be 1 acre.

71. Compare the area of an equilateral triangle with that of the square having one of the sides for its diagonal.

72. The sides of a triangle are 45, 85, 104; find the area.

73. The sides of a triangle are 169, 241, 328; find the area.

74. The sides of a right-angled triangle are 248 ft. and 945 ft. respectively; find the length of the straight line drawn from the right angle to the middle point of the hypotenuse.

75. The sides of a triangle are 20, 21, 29; deduce from the area the length of the perpendicular dropped on to the longest side from the opposite angle, and find the segments of this side made by the perpendicular.

76. In a quadrilateral $ABCD$ the angle at B is a right angle, the sides AB and CD are parallel; AB is 25 ft., CD is 45 ft., and BC is 21 ft. Find the area and the length of the side AD.

77. On opposite sides of a base, which is 110 yards long, two isosceles triangles are constructed, their altitudes being respectively three and four times the base; find in acres the area of the quadrilateral thus formed.

78. The quadrilateral $ABCD$ is inscribed in a circle of which AC is a diameter; if $AB=2$ ft. 4 in., $CD=5$ ft., and $AC=8$ ft. 4 in., find the area of the quadrilateral.

79. The parallel sides of a trapezoid are 47 yds. and 63 yds. respectively, and the area is 1 acre; find the height.

80. Two sides of a trapezoid are parallel and the other two sides are equal; the parallel sides are 10 ft. and 5 ft. respectively, the equal sides are 6 ft. 6 in. each. Find the area.

81. In the last example the two equal sides of the trapezoid are produced to meet, and from the point of intersection a perpendicular is

let fall on the longer of the two parallel sides; find the length of this perpendicular, and show that it is bisected by the shorter of the two parallel sides.

82. ABC is an equilateral triangle, inscribed in a circle; AD is drawn perpendicular to BC and produced to meet the circumference in E. If AB be equal to unity, find the lengths of DE and BE.

83. Two equal circles, the centre of each of which lies on the circumference of the other, intersect; find the length of their common chord, the length of their common radius being unity.

84 Compare the side of a square with that of an equilateral triangle, inscribed in the same circle.

85. In a circle of radius 2 ft. 5 in. two parallel chords are placed, each at a distance of 1 ft. 8 in. from the centre; find the lengths of the chords.

86. Two tangents are drawn to a circle from an external point, and are inclined to one another at an angle of 60°; find the length of their chord of contact, the radius of the circle being unity.

87. In the last example find the length of the lesser arc intercepted by the tangents.

88. The area of a quadrant of a circle is 385 acres; find the radius of the circle $\left(\pi=\frac{22}{7}\right)$.

89. The hypotenuse of a right-angled triangle is 25 ft., and one of the sides is 24 ft.; semicircles are described on the three sides of the triangle. Show that the area of the semicircle described on the hypotenuse is equal to the sum of the areas of the semicircles described on the two sides of the triangle.

90. Two equal circles, the centre of each of which lies on the circumference of the other, intersect. If the radius of each be 1 ft., find the area of the space common to both of them.

91. In travelling from Lahore to Amritsar, a distance of 32 miles, a bicycle wheel makes 23,040 revolutions; find its diameter $\left(\pi=\frac{22}{7}\right)$.

92. Find the area of the circular ring contained between the inscribed and circumscribed circles of a regular hexagon whose side is unity.

93. The radius of a circle is unity; find the radii of the two concentric circles which divide its area into three equal parts.

94. A man who walks at the rate of $2\frac{1}{2}$ miles an hour takes 5 hrs. 8 min. to walk round a circular field; in what time would he walk across it through the centre? $\left(\pi=\frac{22}{7}\right)$.

95. A belt of wood, one mile in width, runs all round the circular base of a hill whose radius is 12 miles; find the length of the longest straight road which can be cut through the wood.

96. A window is in shape a square surmounted by a semicircle; the cost of glazing it at 3a. 6p. per sq. ft. is Rs.4, 14a. Find its width $\left(\pi=\frac{22}{7}\right)$.

97. The radius of a circle is 8 ft. 4 in.; find the area of the space enclosed by the circle and two tangents which intersect at right angles ($\pi=3.1416$).

98. How many yards of canvas, 22 in. wide, will be required for making a conical tent, whose height is 10 ft. 6 in., and radius of base 10 ft.? $\left(\pi=\frac{22}{7}\right)$.

99. The sides of a triangle are 25, 39, and 40 respectively; find the inradius and the circumradius; and find also the segments of the longest side made by the point of contact of the incircle.

100. ABC is a right-angled triangle, the angle C being a right angle. A semicircle is described on AB as diameter and passing through C, and two other semicircles are described on AC and BC as diameters and lying outside the triangle. Show that the sum of the areas, of those portions of the semicircles on AC and BC which lie outside the semicircle on AB, is equal to the area of the triangle ABC.

ANSWERS TO EXERCISES

I. 1. 7 m. 3 dm. 8 cm. 6 mm. 2. 304.3 cm. 3. 465 cm.
4. 273 cm., 505.8 cm., 208.65 cm., 5.5 cm., .7 cm., 40.6 cm., 200.5 cm.
10 500. 11. 10,000. 12. 30.16 inches nearly.

II. 1. 45°, 30°, 60°. 2. 180°, 270°. 3. 70° 18′ 45″.
4 74,250″. 5. $\frac{7291}{21600}$. 6. $\frac{3451}{7200}$. 7. 360°, 252°.
8. 2 right angles. 9. 38°. 10. $82\frac{1}{2}$°.

III. 3. 90°, 1 inch. 4. 90°, 3 cm. 6. 5 cm., 60°, 60°.
7. 60°, 3 inches each. 8. Each side 2 inches, each angle 120°.
9. Angle ACB is 85°.

V. 1. 45°, 45°, 1.41 in. 2. 90°, $BC = AC = 42$ mm.
3. 20 cm., 30°, 60°. 5. 60°, 30°, 3.46 in. 8. Yes.

VI. 1. Yes. 2. Yes. 3. 1 in., 60°. 4. 1 in. 5. 1 in.

VII. 5. Angles are the same, but the sides are twice as long.

VIII. 1. .71 in. 4. One-half of the third side.
7. 60°, 35 mm. 8. Yes.

IX. 1. Yes. 2. Opposite sides and angles are equal.
7. 90°. 9. Yes; each side is one-half of a diagonal.

X. 1. All the angles are right angles, and the diagonals are equal.
2. 45°. 3. Rhombus. 4. Rectangle.
5. Square. 7. Yes; rectangle. 9. Yes.

XI. 1. 8 sq. yds. 5 sq. ft. 2. 28 sq. yds. 8 sq. ft. 96 sq. in.
3. 8 sq. yds. 5 sq. ft. 4. 18 sq. yds. 4 sq. ft. 36 sq. in.
5. 1 sq. yd. 5 sq. ft. 99 sq. in.
6. 94 sq. yds. 3 sq. ft. 104 sq. in.
7. 3161 sq yds. 8 sq. ft. 132 sq. in.
8. 23 sq. yds. $47\frac{1}{2}$ sq. in. 9. 1 sq. yd. 5 sq. ft 9 sq. in.
10. 93 sq. yds. 11. 2361 sq. yds. 6 sq. ft.
12. 1 sq. yd. 6 sq. in. 13. 40 yds.

14. 15 yds. 1 ft. 4 in.   15. 1 yd. $1\frac{1}{2}$ in.
16. 22 yds.   17. 2 yds. 2 ft. 5 in.
18. 1 ft. $\frac{3}{4}$ in.   19. 1 ac. 1.7648 po.
20. 2 ac. 2 ro. 2.5584 po.   21. 1 ac. 23.9264 po.
22. 429 ac. 3 ro. 5.0592 po.   23. 1 sq. yd. 7 sq. ft. 97 sq. in.
24. 34 sq. yds. 6 sq. ft. 16 sq. in.
25. 6 sq. yds. 3 sq. ft. 73 sq. in.
26. 761 sq. yds. 5 sq. ft. $66\frac{1}{4}$ sq. in.
27. 177 sq. yds. 49 sq. in.   28. 8 sq. ft. $108\frac{1}{4}$ sq. in.
29. 1 ac. 1 ro. 37.16 po.   30. 10 ac. 16.04 po.
31. 1 ac. 2 ro. 27.3225 po.   32. 13 ac. 2 ro. 22.7584 po.
33. 1 ac. nearly.   34. 172 ac. 1 ro. 35.8544 po.
35. 59 ft.   36. 101.01 yds.
37. 220 yds.   38. 1 yd. 5.5 in.
39. 5720 yds.   40. 12 ft.   41. 80 yds.
43. 4356 sq. yds., 1936 sq. yds., 484 sq. yds., 484 sq. yds.
44. 99.

XII. 1. 1728.   2. 242.   3. 65.
4. Rs.1140, 7a. 8.16p.   5. 36, £5 : 4 : 6.15d.
6. 20 ft.   7. 3300.   8. $56\frac{8}{9}$.
9. 44.   10. 36.   11. 105.
12. 35.   13. Rs.320.   14. Rs.151, 10a. 8p.
15. £8 : 1 : 11.625d.   16. Rs 223, 7a. $2\frac{2}{15}$p.
17. Rs.220, 2a. 4p.   18. £11 : 5s.   19. Rs.1000.
20. Rs.2115, 3a. 9p.   21. 18 ft.   22. 27 in.
23. 4 p.   24. 100 ft.   25. 25 ft.
26. 1 sq. ch.   27. 9 sq. ft.   28. 56 yds., 40 yds.
29. $35\frac{1}{4}$ ft., $47\frac{2}{3}$ ft., $59\frac{7}{12}$ ft.   30. 4 : 9.
31. 40 ft., 45 ft.   32. 8 in., 5 in.   33. 800 ft.
34. 680 ft.   35. 212.5 ft.   36. 783 ft.
37. 446.4 ft.   38. £6 : 17 : $10\frac{5}{8}$d.
39. £6 : 5 : $7\frac{5}{9}$d.   40. Rs.36, 5a. 4p.
41. £11 : 8 : $9\frac{5}{7}$d.
42. (i.) Rs.13, 6a. $6\frac{9}{14}$p. (ii.) Rs.2, 12a. $9\frac{3}{8}$p. (iii.) Rs.26, 1a. $2\frac{1}{4}$p.
43. 1134 sq. ft.   44. 15,936 sq. yds.
45. Rs.420, 12a.   46. Rs.230, 4a. 8p.
47. 4356 sq. yds.   48. Rs.26, 4a.
49. 13s. $11\frac{1}{3}$d.   50. 6 yds.

XIII. 1. 180°.   7. 59° 51′ 51″.

XIV. 1. 4 right angles.   2. 90°, $1\frac{1}{2}$ in., rhombus.   7. 4 right angles.
8. The exterior angles are together equal to 4 right angles.

XV. 2. 13 cm. 4. 6 in., 3 in.
7. Twice the area of a rectangle whose sides are half the base and half the perpendicular.

XVI. 1. 2 in., 30°.

XVII. 2. Isosceles triangle.

XIX. 2. 5 cm. 3. Equilateral triangles.
4. Each side 3 cm.; each segment 3 cm.
5. Each side 2 in. 6. Yes.

XX. 6. 137 ft. 7. 250 ft. 8. 2306 ft.
9. 1445 ft. 10. 40 ft. 1 in. 11. 4 ft. 10 in.
12. 10 chs. 61 lks. 13. 36 yds. 2 ft. 2 in.
14. 360 ft. 15. 73 yds. 1 ft. 16. 20 yds.
17. 102 ft. 18. 102 yds. 19. 36 yds. 6 in.
20. 125.89 ft. 21. 6.11 yds. 22. 9.67 yds.
23. 142.83 ft. 24. 152 yds. 25. 178 ft.
26. 52 miles. 27. 6 ft. 3 in. 28. 14 ft. 2 in.
29. 83 yds. 5 in. 30. 9.237 ft. 31. 6.928 ft.
32. 20. 33. 31.11 yds.
34. 2 ac. $29\frac{1}{11}$ po. 35. 7 chs. 2 lks. 36. 25 ft.
37. 60 ft. 38. 6.024. 39. 22 ft.
40. 85.44 ft., 72.11 ft.

XXI. 9. 1.1 in. 10. 40°.

XXII. 8. Three.

XXIII. 3. It is a rectangle. 8. Rhombus. 9. 2.2 in.
10. Construct a triangle, base 1.8″, one side 1.4″, and height 1.4″; complete the parallelogram.

XXIV. 3. 243 sq. ft. 4. 14 sq. yds. 96 sq. in.
5. 4 ac. 24.2432 po. 6. 5106 sq. yds.
7. 473 sq. yds. 32 sq. in. 8. 16 yds. 2 ft.
9. 165 yds. 10. 1 ft. 6 in.
11. AB = 110 yds., BC = 176 yds.
12. 30 sq. ft. 13. 4.2426 sq. yds.
14. 17.32 sq. in. 15. 2.598 sq. ft.
16. 3535.5 sq. yds. 17. 36 ac. 1 ro. 8 po.
18. 2165 sq. ft. 19. 150°, 30°.
23. 3.02 sq. in.

XXV. 6. 20 cm. 7. 14 cm., 9.9 cm.
9. In (ii.) the triangle is right-angled. 15. 2 : 1.

XXVI. 1. 83 sq. yds. 5 sq. ft. 72 sq. in. 2. 68.75.
3. 17.590625. 4. 14,280 sq. ft.
5. 9720 sq. ft. 6. 6 ac. 23,520 sq. lks.

7. £200. 8. 39.843 ft.
9. 4800 sq. ft. 10. 360.
11. 756. 12. 504.
13. 306. 14. 336.
15. 990. 16. 14.69.
17. 39.17. 18. 182.97.
19. 1153.21. 20. 465 sq. yds. 1 sq. ft. 126 sq. in.
21. 9 sq. ft. 115 sq. in. 25. 324 yds. nearly.
26. 4 : 1. 27. $12\frac{12}{13}$, 12, $11\frac{1}{5}$.
28. $4 : 3\sqrt{3}$.

XXVII. 1. $1\frac{3}{8}''$, $2\frac{2}{3}''$, $2''$. 2. $\frac{5}{6}''$, $4\frac{1}{6}''$, $1\frac{1}{12}''$.
3. $2\frac{9}{13}$ cm., $4\frac{4}{13}$ cm., $2\frac{23}{26}$ cm.
4. $1\frac{9}{11}$ cm., $3\frac{2}{11}$ cm., $4\frac{4}{11}$ cm.
5. $3''$, $\frac{7}{8}''$, $3\frac{1}{2}''$. 6. $2\frac{5}{14}''$, $3\frac{1}{7}''$, $2\frac{4}{7}''$.
7. $3\frac{1}{6}$ cm., $3\frac{8}{9}$ cm., $4\frac{4}{9}$ cm. 8. $\frac{4}{7}$.
9. $2''$, $2\frac{1}{2}''$, $3''$. 10. Rhombus of side $2\frac{1}{2}''$.
12. $2\frac{2}{3}''$, $2\frac{1}{3}''$, $1\frac{1}{4}''$.

XXIX. 1. $2\frac{2}{5}''$, $3\frac{1}{5}''$, $1\frac{4}{5}''$. 2. $5\frac{7}{13}''$, $\frac{25}{26}''$.
3. $5\frac{19}{28}''$, $\frac{49}{100}''$. 4. $\frac{3}{4}''$, $2\frac{1}{4}''$.

XXX. 1. 6 sq. in., 24 sq. in. 2. 84 sq. ft., 21 sq. ft.
5. $3.3''$, $4.4''$, $5.5''$. 6. 9 · 16.
7. 1 : 4. 8. 1 : 4 : 9, 2 sq in.

XXXI. 1. 1730 sq. ft. 90 sq. in. 2. 29 ac. 1 ro. 24 po.
3. 195 sq. yds. 4. 3 ac., 2 ro. 16 po.
5. 348,040 sq. yds. 6. 8 sq. yds. 6 sq. ft. 90 sq. in.
7. 3750 sq. ft. 8. 730 sq. ft.
9. 294 sq. yds. 10. 27,801.714 sq. yds.
11. 2 ro. 36.02616 po. 12. 163 ac. 3 ro. 22.74 po.
13. 19,959 sq. yds. 2 sq. ft. nearly. 14 252.868 sq. ft.
15. 338 yds., 86,160 sq. yds. 16. 242 ac.
17. 50 ft., 2400 sq. ft. 18. 528 ft. 19. 21 sq. ft.
20. 2420 sq. yds. 21. 6 ft. 22. 160 yds.
23. 75 sq. ft. 24. 60 sq. ft.

XXXII. 1. 80 ac. 2 ro. 12.5 po 2. 17 ac. 1 ro. 11.2 po.
3 14 ac. 7.984 po. 4. 138 ac. 27.5 po.
5. 8 ac. 2 ro. 24.064 po. 6. 8 ac. 2 ro. 7.04 po.
7. 3 ac. 1 ro. 8 po. 8. 1 ac. 3 ro. 21.4 po.
9. 6 ac. 1 ro. 8.8 po. 10. 3 ro 30.8 po.
11. 21 : 25. 12. 2 : 3. 13. Rs.122,857 nearly.

XXXIII. 1. 3 in. 2. 2.4 in. 3 24 cm. 4 1.73 in. 5. 60°.
6. Yes. 7. Yes. 8. The chords are equal. 10. 120°.

XXXIV.	3. 4 in., $4\frac{4}{5}$ in	4. 2.4″, 2″ nearly, $\frac{25}{100}$″.	5. .49″.
XXXV.	4. 2″.		
XXXVI.	1. 15 ft.	2 1 ft. 6 in.	3. 1 ft. 3 in.
	4. 2 ft. 8 in.	5 16 ft.	6 2 ft. 3 in., 3 in.
	7. 2 ft. 1 in.	8. 2 ft.	10. 5.87 in.
	11. 700 sq. ft., or 4900 sq. ft., according as the chords are on the same side, or on opposite sides, of the centre of the circle.		
	12 1 ft 10.62 in.		
XXXVII.	1. 4 in.	2. 2 ft. 9.8 in.	3. 8 ft.
	4. 5.25.	5. 1.63.	6. 2.65.
	7 .31.	8. 4.77.	9. 2.88.
	10. 1.54.	14 9 in.	16. 29,160 sq. yds.
XXXVIII.	5. 106 ft.	6. 32.5.	7. 21.0125.
	8. $30.0\dot{3}$.		9. 127.5.
	11. 4 yds. 2 ft. 1.04 in.		13. 3 : 8
XXXIX.	1. 1 ft.	2. 259 8 sq. ft.	3. 5.196 yds.
	4. 1.299 sq. yds.	5. 3 46 sq. in.	6. 3 : 4.
	7. 6.21 in.	8. 3 sq yds.	9. $2 : \sqrt{3}$.
	10. 1131.3 sq. ft.	11. Rs 2244, 9a. 9p.	12. 43 yds. 6 in.
	13. 3 : 1.	14. 26 sq. ft. 114 sq. in.	
XL	1. 1 yd 1.699 in.	2. 100.007 ft.	
	3. 8.028 yds.	4. 15.969 yds.	
	5. 31,416 yds.	6. 29.321 yds.	
	7. 2 ft. 7 in.	8. 1 ft. 9 in.	
	9. 38 ft. 9.5 in.	10. 3 yds. 2 ft. 4.5 in.	
	11. 89 ft. 10 in.	12. 9 ft. 11 in.	
	13. 16 ft. 6 in.	14. 8 yds. 1 ft. 8 in.	
	15. 15.91 ft.	16. 280.11 yds.	
	17. 1 yd. 2 ft. .4 in.	18 $4844.\dot{4}$.	
	19. 71,530.	20. 144,463.35 sq. yds.	
	21. 1 ft. 9 in.		
XLI.	1. 1134 sq. ft. 17 sq. in.	2. 49,087.5 sq. ft.	
	3. 795.72 sq. ft.	4. 1 ch.	
	5. 78.5 yds.	6. 28.2744 sq. ft.	
	7. 1074 577 sq. yds.	8. 8.38 yds	
	9. 17 ft. 5 in.	10. 50.2656 sq. ft.	
	12. 25 sq. ft.	13. 911.3258 sq. ft.	
	14. 36°.	15. .8584 sq. in.	
	16. 2 0944 sq. ft.	17. 74.01 sq. ft., 4.53 sq. ft	
	18. 5.63 sq. ft.	19. $31\frac{3}{7}$.	20. 1.0272 sq. in.

MISCELLANEOUS EXAMPLES

51. 22 yds.	52. 363 ghumaons.	53. 1000.
54. 40 ft.	55. 370 ft.	56. 9792 sq. yds.
57. 3120.5 sq. yds.	58. 40,245 sq. yds. 3 sq. ft.	
59. Rs.5, 10a. 3p.	60. 440 yds.	61. 6 in.
62. 25 ft.	63. 1005 sq. ft. ; 15a. 1p. nearly.	
64 15 ft.	65. 805 sq. ft.	66 120.
68 360 ft.	69. 58 ft.	70. 110 yds.
71 $\sqrt{3}:2$.	72. 1872.	73. 19,680.
74 488 ft. 6 in.	75. $14\frac{14}{25}$, $13\frac{23}{25}$, $15\frac{6}{25}$.	
76. 735 sq. ft. ; 29 ft.		77. $8\frac{3}{4}$ acres.
78. 17 sq. ft. 96 sq. in.		79. 88 yds.
80. 45 sq. ft.	81. 12 ft.	82. $\frac{\sqrt{3}}{6}$, $\frac{\sqrt{3}}{3}$.
83. $\sqrt{3}$.	84. $\sqrt{2}:\sqrt{3}$.	85. 42 in. each.
86. $\sqrt{3}$.	87. $\frac{2\pi}{3}$.	88. 1540 yds.
90 $\frac{1}{6}(4\pi-3\sqrt{3})$.	91. 2 ft. 4 in.	92. $\frac{\pi}{4}$.
93 $\sqrt{\frac{1}{3}}$, $\sqrt{\frac{2}{3}}$.	94. 1 hr. 38 min.	95. 10 miles.
96. 4 ft.	97. 14 sq. ft. 130 sq. in.	
98. $745\frac{5}{7}$.	99. 9, $20\frac{5}{6}$; 13, 27.	

THE END

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